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DEVOTED TO THE
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PAPERS ON MATHEMATICAL SUBJECTS, BIOGRAPHIES
OF NOTED MATHEMATICIANS, ETC.

EDITED BY

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AUTHOR OF FINKEL'S MATHEMATICAL SOLUTION BOOK, MEMBER OF THE AMERICAN MATHEMATICAL SOCIETY, AND PROFESSOR OF MATHEMATICS AND PHYSICS IN DRURY COLLEGE, SPRINGFIELD, MISSOURI.

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BOLYAI FARKAS. [WOLFGANG BOLYAI.]

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. III. JANUARY, 1896. No. 1.

BIOGRAPHY.

BOLYAI FARKAS. [WOLFGANG BOLYAI.]

BY DR. GEORGE BRUCE HALSTED.

FOR the treatment of parallels, what Frischauf calls "das anschaulichste Axiom," is due to the researches of Bolyai Farkas. He gives it in his "Kurzer Grundriss eines Versuchs" etc., p. 46, as follows: "Koennten jede 3 Punkte, die nicht in einer Geraden sind, in eine Sphaere fallen; so waere das Eucl. Ax. XI. bewiesen." Thus the space whose every three points are co-straight or concyclic is Euclidean.

But in his Autobiography written in Magyar, of which my forthcoming life of the Bolyais contains the first translation ever made, he says: "Yet I was not satisfied with my attempts to prove the Problem of Parallels, which was ascribable to the long discontinuance of my studies, or more probably it was due to myself that I drove this problem to the point which robbed my rest, deprived me of tranquility."

Hitherto what was known of the Bolyais came wholly from the published works of the father, Bolyai Farkas, and from a brief article by Architect Fr. Schmidt of Budapest, "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya. Grunerts Archiv, Bd. 48, 1868, p. 217.

In two communications sent me in September and October, 1895, Herr Schmidt has very kindly and graciously put at my disposal the results of his subsequent researches which I will here reproduce. But meantime I have from entirely another source come most unexpectedly into possession of original documents so extensive, so precious that I have determined to issue them in a

separate volume devoted wholly to the life of the Bolyais ; but these are not used in the sketch here given.

Bolyai Farkas was born February 9th, 1775, at Bolya in that part of Transylvania (Erdély) called Székelyföld. He studied first at Enyed, afterward at Klausenburg (Kolozsvár), then went with Baron Simon Kemény to Jena and afterward to Goettingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life.

The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss. Everyone who met Bolyai felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written in 1801 that Bolyai was one of the most extraordinary men he had ever known.

He returned home in 1799, and in January, 1804, was made professor of mathematics in the Reformed College of Maros-Vásárhely. Here for 47 years of active teaching he had for scholars nearly all the professors and nobility of the next generation in Erdély.

Sylvester has said that mathematics is poesy.

Bolyai's first published works were dramas.

His first published book on mathematics was an arithmetic : Az arithmetica eleje. 8vo. I—XVI, 1—162 pp. The copy in the library of the Reformed College is enriched with notes by Bolyai János.

Next followed his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, 8vo. with title as follows : TENTAMEN | JUVENTUTEM STUDIOSAM | IN ELEMENTA MATHESEOS PURÆ, ELEMENTARIS AC | SUBLIMIORIS, METHEDO INTUITIVA, EVIDENTIA— | QUE HUIC PROPRIA, INTRODUCENDI. | CUM APPENDICE TRIPLICE. |

Auctore Professore Matheseos et Physices Chemiæque | Publ. Ordinario. | Tomus Primus. | *Maros Vásárhelyini.* 1832. | Typis Collegii Reformatorum per JOSEPHUM, et | SIMEONEM KALI de felsö Vist. | At the back of the title : Imprimatur. | M. Vásárhelyini Die | 12 Octobris 1829. |

The now world renowned Appendix by Bolyai János was an afterthought of the father, who prompted the son not 'to occupy himself with the theory of parallels,' as Staeckel says, but to translate from the German into Latin a condensation of his treatise, of which the principles were discovered and properly appreciated in 1823, and which was given in writing to J. W. von Eckwehr in 1825.

The father, without waiting for Vol. II., inserted this Latin translation, with separate paging (1—26), as an Appendix to his Vol. I., where, counting a page for the title and a page 'Explicatio signorum,' it has twenty-six numbered pages, followed by two unnumbered pages of Errata.

The treatise itself, therefore, contains only twenty-four pages—the most extraordinary two dozen pages in the whole history of thought !

Milton received but a paltry 5 pounds for his Paradise Lost ; but it was at least plus 5. Bolyai Janos, as we learn from Vol. II., p. 384 of '*Tentamen*,' contributed for the printing of his eternal 26 pages, 104 florins 54 kreuzers.

That this Appendix was finished considerably before the Vol. I., which it follows, is seen from the references in the text, breathing a just admiration for the Appendix and the genius of its author,

Thus Bolyai Farkas says, p. 452: Elegans est conceptus *similium*, quem J. B. *Appendicis Auctor* dedit; again, p. 489: *Appendicis Auctor*, rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit; quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis. And the volume ends as follows, p. 502: Nec operae pretium est plura referre; quum res tota ex altiori contemplationis puncto, in ima prenetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno digna legi.

The father gives a brief resumé of the results of his own determined, life-long, desperate efforts to do that at which Saccheri, J. H. Lambert, Gauss also had failed, to establish Euclid's theory of parallels *a priori*.

He says, p. 490: "tentamina idcirco quae olim feceram, breviter exponenda veniunt; ne saltem alius quis operam eandem perdat." He anticipates J. Delboeuf's "Prolégomènes philosophiques de la géométrie et solution des postulato," with the full consciousness in addition that it is *not* the solution,—that the final solution has crowned not his own intense efforts, but the genius of his son.

This son's Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriprive adjective, *Euclidean*, this wonderful production of pure genius, this strange Hungarian flower was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. He it was who first did justice publicly to the works of Lobachevski and Bolyai.

Incited by Baltzer, 1866, J. Hoüel issued a French translation of Lobachevski's Theory of Parallels and in a note to his Preface says: "M. Richard Baltzer, dans la seconde édition de ses excellents *Éléments de Géométrie*, a, le premier, introduit ces notions exactes à la place qu'elles doivent occuper." Honor to Baltzer! But alas! father and son were already in their graves!

Fr. Schmidt in the article cited (1868) says: "It was nearly forty years before these profound views were rescued from oblivion, and Dr. R. Baltzer, of Dresden, has acquired imperishable titles to the gratitude of all friends of science as the first to draw attention to the works of Bolyai, in the second edition of his excellent Elemente der Mathematik (1866-67). Following the steps of Baltzer, Professor Hoüel, of Bordeaux, in a brochure entitled: Essai critique sur les principes fondamentaux de la Géometrie élémentaire, has give extracts from Bolyai's book, which will help in securing for these new ideas the justice they merit."

The father refers to the son's Appendix again in a subsequent book, Ürtan elemei Kezdöknek [Elements of the science of space for beginners] (1850-51), pp. 48. In the College are preserved three sets of figures for this book, two by the

author, and one by his grandson, a son of János. The last work of Bolyai Farkas, the only one composed in German, is entitled: Kurzer Grundriss eines Versuchs

I. Die Arithmetik, durch zvekmässig Konstruirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit u.d.gl. nicht nur scharf zu bestimmen; sondern auch ihr Seyn im Raume zu beweisen: und da die Frage, *ob zwey von der dritten geschnittene Geraden, wenn die summe der inneren Winkel nicht* $=2R$, *sich schneiden oder nicht?* Niemand auf der Erde ohne ein Axiom (wie Euklid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern; und eine auf die *Ja*-Antwort, andere auf das *Nein* so zu bauen, das die Formeln der letzten, auf ein Wink auch in der ersten gültig seyen.

Nach ein lateinischen Werke von 1829, M. Vásárhely, und eben daselbst gedruckten ungrischen.

Maros Vásárhely 1851. 8vo. pp. 88.

In this book he says, referring to his son's Appendix: "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . . . From Goettingen the giant of mathematics, who from his pinnacle embraces in the same view the stars and the abysses, wrote that he was surprised to see accomplished, what he had begun, only to leave it behind in his papers." This refers to 1832. The only other record that Gauss ever mentioned the book is a letter from Gerling written October 31st, 1851, to Wolfgang Bolyai on receipt of a copy of 'Kurzer Grundriss.' Gerling, a scholar of Gauss, had been from 1817 Professor of Astronomy at Marburg. He writes: "I do not mention my earlier occupation with the theory of parallels, for already in the year 1810—1812 with Gauss, as earlier as 1809 with J. F. Pfaff I had learned to perceive, how all previous attempts to prove the Euclidean axiom had miscarried. I had then also obtained preliminary knowledge of your works, and so, when I first [1820] had to print something of my view thereon, wrote it exactly so, as it yet stands to read on page 187 of the latest edition.

We had about this time [1819] here a law professor Schweikart, who was formerly in Charkow, and had attained to similar ideas, since without help of the Euclidean axiom he developed in its beginnings a geometry which he called Astralgeometry. What he communicated to me thereon, I sent to Gauss, who then informed me, how much farther already had been attained on this way and later also expressed himself about the great acquisition, which is offered to the few expert judges in the Appendix to your book."

The 'latest edition' mentioned appeared in 1851, and the passage refarred to is: "This proof [of the parallel-axiom] has been sought in manifold ways by acute mathematicians, but yet until now not found with complete sufficiency. So long as it fails, the theorem, as all founded on it, remains a hypothesis, whose

validity for our life indeed is sufficiently proven by *experience*, whose *general, necessary exactness* however could be doubted without absurdity."

Alas! that this feeble utterance should have seemed sufficient for more than thirty years to the associate of Gauss and Schweikart, the latter certainly one of the independent discoverers of the non-Euclidean geometry. But then since neither of these sufficiently realized the transcendent importance of the matter to publish any of their thoughts on the subject, a more adequate conception of the issues at stake could scarcely be expected of the scholar and colleague. How different with Bolyai János and Lobachévski, who claimed at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

Austin, Texas, December 16th, 1895.

THE DUPLICATION OF THE NOTATION FOR IRRATIONALS.

By JOS. V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

Authorities agree in crediting Rudolff (1525), the German cossist, with the introduction of the radical sign, $\sqrt{}$, not precisely as we use it, but one such mark for a square root, three for a cube, and two for a fourth root. Cantor thinks it probably originated from a West-Arabian custom of using dots, by makings *lines* of the dots, and connecting them in the making by lighter lines. These dots in turn originated, it is thought, in the use of the letter, dschim, the first in the Arabian word for *root*. Rudolff was followed by Stifel in the employment of this notation, and afterwards Girard (1633) changed it to the present form. By the middle of the 17th century the mark had come into general use. The exponent notation, though first used by Rudolff and Stifel in a crude form, was employed as we now have it for integral values of the exponents by Descartes. Soon after, Wallis, in his *arithmetica infinitorum* (1656), interpreted and used the simpler forms of fractional exponents, though Stevin (1585) had suggested the meaning to be assigned them. Then in 1676 Newton wrote to Oldenburg "since algebraists write a^2, a^3, a^4, etc., for *aa*, *aaa*, *aaaa*, etc., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{e.a^5}$." Newton went further in connection with his binomial theorem, and generalized this use of exponents into the exponential function. The question naturally arises why was it that the old notation for roots was not replaced by the new as had been done in numerous instances before? Doubtless the best

reason for this is the fact that the radical signs were firmly intrenched by extended use before the fractional exponents as we have them were even thought of.

Now from one standpoint at least this duplication of marks for one of the commonest operations in mathematics is unfortunate. It certainly complicates unnecessarily a rather difficult part of elementary algebra. Doubtless all would agree that one or the other should be given up unless there is a good and sufficient reason for its retention. If either is to be discarded there is no question for a moment as to which should go. The use of fractional exponents is in perfect accord with that of integral ones, and introduces no new marks or conventions, while the radical sign notation is out of harmony with everything else in the algebraic notation. The radical sign and index are new marks, while the fractional exponent is an old quantity in a new place whose interpretation is quite natural. However, it should be said that the fractional exponent notation is ambiguous, since, in general,

$\left(a^m\right)^{\frac{1}{n}}$ will not be the same as $\left(a^{\frac{1}{n}}\right)^m$, though each reduces to $a^{\frac{m}{n}}$. Nevertheless, even here the fractional exponent notation is to be preferred to the others, since the elementary treatment of irrationals virtually depends on the ignoring of this difference. (See, for example, Todhunter's Algebra, ed. 1877, p. 153; Chrystal's Treatise, Chapter X, Part II.) Not a few authors succeed by their manner of treatment in slurring this over. In this connection it ought to be said that some authors' books show distinct traces of their having been confused by the double surd notation. If authors themselves are not clear in their treatment of irrationals, it is likely that their students also will be more or less puzzled. This of itself would be a sufficient justification of an effort to remove the difficulty.

One obstacle in the way of dispensing entirely with the radical signs consists in the practical difficulty of writing and printing fractional exponents. But this, one is constrained to believe, can readily be overcome. And first it may be remarked that there is the same justification for omitting the numerator 1 in a fractional exponent that there is for never writing the integral exponent 1. When omitted it can be understood. Then again there is the same justification for dropping the denominator 2 in the exponent that there is for understanding the radical index 2 when no index is written. Thus all that is left of the fractional exponent $\frac{1}{2}$ is the horizontal line or the solidus oblique line. To make the changes suggested clear to the reader, some expressions are written below with their values in the three notations:

RADICAL NOTATION.		FRAC. EXPONENT NOT.		PROPOSED NOTATION.
$2\sqrt{a}$	$=$	$2a^{\frac{1}{2}}$	$=$	$2a^{\prime}$*

*The marks for primes would differ from this sign in being shorter and vertical. However, it would be better to write subscripts in place of them.

$$
\begin{array}{cccc}
3\sqrt[3]{26} & = & 3(26)^{3} & = & 3(26)^{/3} \\
\sqrt{a+m} & = & (a+m)^{\frac{1}{2}} & = & (a+m)^{/} \\
\sqrt[4]{\dfrac{a^2+b^2+c^2}{2abc}} & = & \left(\dfrac{a^2+b^2+c^2}{2abc}\right)^{\frac{1}{4}} & = & \left(\dfrac{a^2+b^2+c^2}{2abc}\right)^{/4} \\
\sqrt[5]{(x^3+3xy^2)^3} & = & (x^3+3xy^2)^{\frac{3}{5}}\text{, or } (x^3+3xy^2)^{3/5} & &
\end{array}
$$

The proposed notation would do away with vinculums and would use preferably the solidus sign for division as is the tendency now in English mathematical and scientific books. In printing, $\sqrt{}$ would be replaced by $)^{/}$ on one type, and in script the latter would be made, without lifting the pen, in loop form. However, when the numerator of the fractional exponent is other than unity, the usual fractional exponent notation (which for this case is preferable to the radical sign notation) would be employed. Notice that by the simple changes proposed, which are perfectly natural ones, all the advantages of the duplicate notation would be preserved with none of its disadvantages, such as the use of the unsightly hieroglyphic-like radical sign (giving as it does a forbidding appearance to the printed page), and the confusion which arises from the simultaneous use of two distinct notations for the same operation.

In conclusion it should be emphasized that mathematicians themselves are not likely to feel the need or approve of any change in the algebraic notation. Like the reform in spelling, it is in the interest chiefly of the hundreds of thousands of students of elementary mathematics yet to come, and not in that of those who have already mastered the two notations, that this reform is urged. Surely it is not too much to ask that the fractional exponents as now written be employed exclusively (instead of largely as now) in all higher works involving the use of algebraical symbols. The abridgments would then be likely to come as a matter of course.

Stevens Point, Wisconsin, May 11, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from December Number.]

THE CONSTRUCTION OF NON-PRIMITIVE GROUPS WITH THREE SYSTEMS OF NON-PRIMITIVITY.

Let the degree of the required group G be $3n$. G must be a subgroup (using subgroup in its broad sense in wnich it includes the group itself and identity) of

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}.$$

If G_1 is not identity,* its constituents must be conjugate transitive subgroups of these three systems.

If we designate the systems by A, B, and C, the permutations of the systems must correspond to a group of these three letters, for if these permutations would not form a group of operations G itself could not be a group. Hence every non-primitive group with three systems must correspond to one of the following groups:

$$(ABC)\text{cyc} \qquad (ABC)\text{all}$$

Since the former of these is a subgroup of the latter it follows that at least a part of every non-primitive group in three systems corresponds to

$$(ABC)\text{cyc}$$

we proceed to find this part. By a course of reasoning similar to that employed under two systems it follows that all the substitutions which transform any G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 b_1 c_1 . a_2 b_2 c_2 \ldots\ldots a_n b_n c_n$$

and all those which transform G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 c_1 b_1 . a_2 c_2 b_2 \ldots\ldots a_n c_n b_n$$

These sets are not independent, for if

$$s_\gamma \qquad \gamma = 1, 2, \ldots\ldots (n!)^3$$

represents the substitution of one set then will the $(n!)^3$ different corresponding values of

$$s_\gamma^{-1}$$

represent the substitutions of the other set.

If in any non-primitive group G_2 stands for the substitution belonging to the first set and G_3 for those belonging to the second set, and if g_2 and g_3 represent the number of substitutions in G_2 and G_3 respectively we derive from the fact that if a group contains s_γ it must also contain s_γ^{-1} that

$$g_2 = g_3$$

If in any non-primitive group we multiply any substitution of G_2 by all

*This case was not considered under two systems of non-primitivity. It was unnecessary to consider it. For, since a transitive group contains substitutions which replace a given letter by all of the letters involved it follows that the order of a non-primitive group is always equal to its degree. It can easily be shown that the order of any transitive group is a multiple of its degree.

the substitutions of G_3 we obtain g_3 different substitutions of G_1, hence

$$g_1 \geqq g_3.$$

If we multiply a given substitution of G_3 into all the substitutions of G_1 we obtain g_1 different substitutions of G_3, hence

$$g_3 \geqq g_1.$$

Combining the last two relations with the preceding we obtain for any non-primitive group with three systems of non-primitivity

$$g_1 = g_2 = g_3.$$

Since the relation between G_2 and G_3 is such that we can derive one directly from the other we shall generally consider only G_2. But G_2 can be directly obtained from G_1 provided we have given one of the substitutions of G_2. Hence to construct the non-primitive group (or the part of a non-primitive group) corresponding to

$$(ABC)\text{cyc}$$

it is only necessary to find G_1 and one substitution (s_γ) corresponding to ABC.

s_γ must clearly satisfy the following conditions:

(1) Its cube is found in G_1.

(2) It transforms G_1 into itself.

(3) It permutes the systems according to ABC.

These three conditions are sufficient for if any substitution s_γ fulfills these conditions then is

$$G_1 + G_1 s_\gamma + G_1 s_\gamma^{-1}$$

a non-primitive group for

$$G_1 s_{\gamma_1} G_1 = G_1 s_{\gamma_1} G_1 s_\gamma^{-1} s_{\gamma_1} = G_1 s_{\gamma_1}$$

$$G_1 s_\gamma^{-1} G_1 - G_1 s_\gamma^{-1} G_1 s_{\gamma_1} s_\gamma^{-1} = G_1 s_\gamma^{-1}$$

etc., etc., etc.

It remains to prove that the three given conditions are necessary as well as sufficient, i. e., we have to show that none of the three pair of conditions is sufficient. The pair which excludes the last condition is evidently insufficient, and the following examples prove that the other two pair are also insufficient.

1	1	1	1
abc	*def*	*ghi*	*abc.def.ghi*
acb	*dfe*	*gih*	*acb.dfe.gih*
			ab.de.gh
			ac.df.gi
			bc.ef.hi

For $aehbdg.cfi$ satisfies the second and third but not the first of the three conditions if we take the first of these groups for G_1, and $aehbficdg$ satisfies the first and third but not the second if we take the second of these groups for G_1. Hence we see that the three given conditions are necessary as well as sufficient.

If the transitive constituents of G_1 admit only a cyclical (not a symmetric) permutation then it is impossible to construct a G corresponding to (ABC)all and involving the given G_1. If they admit a symmetric permutation we have to add to the part of G corresponding to (ABC)cyc sufficient substitutions to make it correspond to (ABC)all. By a course of reasoning similar to that which we have just pursued we prove that it is only necessary to find one substitution s_β corresponding to AB, and that s_β must satisfy the following conditions:

(1) Interchange the first two systems.

(2) Have its square in G_1.

(3) Transform the group corresponding to ABC into itself.

To fix these ideas we proceed to the construction of the non-primitive groups of degree six which contain three systems of non-primitivity. We shall then have found all the non-primitive groups up to degree eight as no such groups can exist for degree seven, or any other prime degree.

Non-Primitive Groups of Degree Six with Three Systems of Non-Primitivity.

G_1 must be one of the following four groups: $(ab)(cd)(ef)$, $\{(ab)(cd)(ef)\}$pos, $(ab.cd.ef)$, 1 G_2 must be contained in

$$(ab)(cd)(ef)\ ace.bdf$$

(*a*) If $G_1=(ab)(cd)(ef)$ then will $ace.bdf$ evidently satisfy the three necessary conditions, we thus obtain a non-primitive group corresponding to ABC, whose order is 24, viz:

(1) $$(ab)(cd)(ef)\ (ace.bdf)\text{cyc}=(abcdef)_{24}*$$

For s_β we may take $ac.bd$. This leads to a group of order 48 which has the preceding group as a self-conjugate sub-group. The group is

(2) $$(ab)(cd)(ef)(ace.bdf)\text{cyc}(ac.bd)=(abcdef)_{48}$$

(*b*) If $G_1=\{(ab)(cd)(ef)\}$pos we can again use $ace.bdf$ for s_γ. We thus obtain a second non-primitive group of order 12, viz:

(3) $$\{(ab)(cd)(ef)\}\text{pos}\ (ace.bdf)=(abcdef)_{12}\ \dagger$$

This is the only group that corresponds to ABC since the negative substitutions which correspond to the most general G_2 do not have their cubes in this

*The foot note in regard to $(abcdef)_{24_2}$ applies also to this group.

†The foot note in regard to $(abcdef)_{24_2}$ applies also to this group.

G_1. For s_β we may take both *ac.bd* and *adbc*. We thus obtain two additional groups of order 24, viz:

(4) $\{(ab)(cd)(ef)\}$pos$(ace.bdf)(ac.bd)=(+abcd)_{24}$

(5) $\{(ab)(cd)(ef)\}$pos $(ace.bdf)(adbc)=(\pm abcd)_{24}$

(*c*) If $G_1=(ab.cd.ef)$, s_γ may again equal *ace.bdf*. The two substitutions *ab.cd.ef* and *ace.bdf* generate the group. The first interchanges the two cycles of the second and the second interchanges the three cycles of the first. The resulting group must therefore have two as well as three systems of non-primitivity, and hence is found in the former list. All the other three possible groups corresponding to *ABC* are conjugate to this.

For s_β we may use *ac.bd*, but with *ace.bdf* this will generate $(ace.bdf)$all. Hence this group is also found in the list of non-primitive groups with two systems of non-primitivity. Hence there is no additional non-primitive group for $G_1=(ab.cd.ef)$.

(*d*) If $G_1=1$ the second condition of s_γ is satisfied by every substitution. The substitutions that may correspond to *ABC* must be of the third order and are therefore all conjugate so that we need to consider only one of them. We thus obtain the intransitive group

$$(ace.bdf)\text{cyc}.$$

If we take *ac.bd* for s_β we obtain an intransitive group corresponding to (ABC)all. If we take *ab.dc.ef* for s_β we obtain a non-primitive group which is also non-primitive in two systems as is evident. Hence $G_1=1$ leads to no new non-primitive group.

We have now examined the entire region through degree six with a view to its non-primitive groups and have found the following

List of Non-Primitive Groups Through Degree Six.

Degree	Order	No.	Group
4	4	1	$(abcd)_4$
		2	$(abcd)$cyc
	8	1	$(abcd)_8$
6	6	1	$(abcdef)_6$
		2	$(abcdef)$cyc
	12	1	$(abcdef)_{12}$
		2	$(abcdef)_{12'}$
	18	1	$(abcdef)_{18}$
	24	1	$(+abcdef)_{24}$
		2	$(\pm abcdef)_{24}$
		3	$(abcdef)_{24'}$
	36	1	$(abcdef)_{36}$

	2	$(abcdef)_{36}$,
48	1	$(abcdef)_{48}$
72	1	$(abcdef)_{72}$

GENERAL REMARKS ON THE CONSTRUCTION OF NON-PRIMITIVE GROUPS.

Let it be required to find the non-primitive groups of degree n, n being a composite positive integer greater than three, and let

$$m_1, m_2, \quad . \quad . \quad . \quad m_e$$

be all the positive integral factors of n (excepting unity) which satisfy the relation

$$m\alpha \leqq \sqrt{n} \qquad \begin{array}{l} \alpha = 1, 2, \quad . \quad . \quad . \quad e \\ \sqrt{\ } \text{ indicates only the arithmetic root.} \end{array}$$

Hence we may divide n as follows:

No. of Systems	No. of Letters in Each System
m_1	$\frac{n}{m_1}$
m_2	$\frac{n}{m_2}$
.	.
.	.
.	.
m_e	$\frac{n}{m_e}$
$\frac{n}{m_1}$	m_1
$\frac{n}{m_2}$	m_2
.	.
.	.
.	.
$\frac{n}{m_e}$	m_e

Two of these relations will become identical when $m_\alpha = \sqrt{n}$ for some value of α in the series

$$1, 2, \quad . \quad . \quad . \quad e.$$

Otherwise they will all be different. From these we see that the number of different ways of dividing n into systems is odd or even as n is or is not a perfect square.

The work of finding all the non-primitive groups for any one of these divisions into systems, *e. g.* the one which contains m_1 systems, may be resolved into the following steps:

(1) Contruct the groups (the G_1's) which have conjugate transitive constituent groups from each of these systems and are so constituted that their con-

stituents admit of the permutations of some transitive group of degree m_1. The constituent transitive groups are clearly of degree $\frac{n}{m_1}$ unless $G_1=1$. The last case does not need consideration when the order of the transitive group of degree m_1 is not a multiple of n.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from December Number.]

SCHOLION IV: *In which is expounded on a figure a certain consideration on which Euclid probably thought, in order to establish that Postulate of his as 'per se' evident.*

I premise first: within any acute angle BAX (Fig. 12.) can be drawn from any point X of AX a certain straight XB, which under designated even if obtuse angle R, which only with this acute BAX falls short of two right angles; a certain XB, say I, can be drawn, which at a finite remove meets this AB in a certain point B. For just that I have demonstrated in a Scholion after P. XIII. I premise secondly: these AB, AX (Fig. 25) can be understood as produced into the infinite even to certain points Y, and Z; and likewise the aforesaid XB (into the infinite and itself produced even to a certain point Y) can be understood to be so moved above this AB toward the parts of the point Z, that the angle at the point X toward the parts of the point A is always equal to the certain given obtuse angle R.

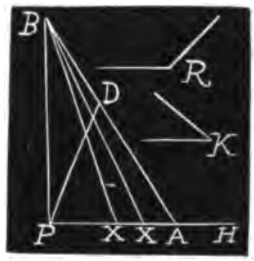


Fig. 12.

Fig. 25.

I premise thirdly: that Euclidean Postulate would be liable now to no doubt, if the aforesaid XY in this however great motion above the straight AZ cuts always that AY in certain points B, D, H, P, and so successively in other points more remote from this point A.

The reason is evident; since thus any two straights AB, XH lying in the same plane, upon which any straight AX cutting makes two angles toward the

same parts BAX, HXA, less than two right angles, must at length meet toward those parts in one and the same point H.

I premise fourthly: likewise will be no doubt over the truth of the preceding hypothetical assumption, if those later external angles YHD, YDP and so any other succeeding ones, either always are equal to the preceding external angle YBD, or at least always will be not so much less but that any one of them always will be greater than any little designated acute angle K. For, this holding, it is manifest that this XY in that however great motion of its toward the parts of the point Z, never will cease to cut the aforesaid AY; which assuredly (from the preceding note) is sufficient for establishing the controverted postulate.

Solely therefore remains, that a certain adversary may say those external angles at greater and greater distance from that point A may become always less without any determinate limit.

But thence would follow, that that XY in that motion of its above the straight AZ would at length meet AY in a certain point P without any angle with the segment PY, so that indeed a segment of the two straights APY, and XPY would be in this way common.

But this is evidently repugnant to the nature of the straight line. [The possibility that P may be a point at infinity is here overlooked.]

But if indeed to anyone may seem less opportune the obtuse angle at that point X toward the parts of the point A, it may easily be supposed right; so that indeed (in the motion of the aforesaid XY at angles always right above the straight AZ) more manifestly may appear that the single points of that XY are always moved equably relatively to the basal AZ; and therefore the aforesaid XY cannot go over from a secant into a non-secant of the other indefinite AY, unless either once in some point it precisely touches it, or meets it in some point P, where it has with this AY a common segment PY; each of which I will show contrary to the nature of the straight line in P. XXXIII.

Therefore in accordance with the true idea of the straight line, must that XY, in however great distance of the point X from the point A, always meet in some point this AY. And that this indeed (however small is supposed the acute angle at the point A) is sufficient for demonstrating, against the hypothesis of acute angle, the Euclidean Postulate, will follow from P. XXVII.

[To be Continued.]

THE BOND PROBLEM.

By J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pennsylvania.

What should an investor pay for one 7 per cent. $100. bond to run 20 years, interest payable semi-annually, in order to realize 8 per cent. per annum, payable semi-annually?

Let X=the price paid; R=4%, the semi-annual rate the investor realizes; t=the whole number of interest payments; $r=3\frac{1}{2}\%$, the rate the bond draws semi-annually; v=$100.

Besides the interest, the investor gains $v-x$, which will be due in $\frac{1}{2}t$ years. To liquidate both of these by equal payments requires each semi-annual payment to include the interest (rv) and such portion of the discount ($v-x$) as would, compounded semi-annually at $R\%$, amount to $v-X$ in $\frac{1}{2}t$ years.

Let y be such a sum; then

$y(1+R)^{t-1}$=amount of 1st installment at end of $\frac{1}{2}t$ years.
$y(1+R)^{t-2}$= " " 2nd " " " " "
$y(1+R)$= " " $(t-1)^{th}$ " " " " "
$y(1+R)^0$ = " " t^{th} " " " " "

Hence, $y[(1+R)^{t-1}+(1+R)^{t-2}+\ldots\ldots(1+R)+1]=v-X.$

Summing the geometrical progression within the brackets, we have

$$y\left[\frac{(1+R)^t-1}{R}\right]=v-X,$$

$$\text{whence } y=\frac{R(v-X)}{(1+R)^t-1}.$$

Therefore each of the t equal payments is

$$vr+\frac{R(v-X)}{(1+R)^t-1},$$

which divided by X gives R.

$$\text{Hence, } vr+\frac{R(v-X)}{(1+R)^t-1}=RX.$$

Solve this equation for X and we have:

$$X=\frac{v(R-r)+vr(1+R)^t}{R(1+R)^t}\ldots\ldots(A).$$

In the above general equation substitute values from the problem and we have:

$$X=\frac{100(.04-.03\frac{1}{2})+3\frac{1}{2}(1.04)^{40}}{.04(1.04)^{40}}=\frac{\frac{1}{2}+3\frac{1}{2}\times1.04^{40}}{.04\times1.04^{40}}.$$

The easy numerical computations are as follows:

40log 1.04=0.0170333 × 40=0.681332, which corresponds to 4.801.

$$\frac{\frac{1}{2}+3\frac{1}{2}\times4.801}{.04\times4.801}=\frac{17.3035}{.19204}=90.1036.$$

$\therefore$ \$90.1036 is the price to be paid for a 7% \$100 bond, interest payable semi-annually for 20 years, in order to realize 8% per annum, payable semi-annually.

The general equation (A) can be applied to the solution of the quarterly bond. In so applying it "we solve the government problem which confronted the Secretary of the Treasury when he placed the late \$50,000,000 loan on the market." This problem has been admirably solved by Theodore L. DeLand, the distinguished Examiner of the U. S. Civil Service Commission, first by algebraic analysis in The American Mathematical Monthly, and later by using the Calculus of Finite Differences. The latter solution was issued under cover of the *Mathematical Magazine*, January, 1895.

Secretary Carlisle desired to sell 10-year 5% \$100 bonds, interest payable quarterly, at a price that would enable the purchaser to realize 3%, interest payable quarterly.

Using these data, we have $R=\frac{3}{4}\%$, $r=1\frac{1}{4}\%$, $t=40$. Substituting values, equation (A) becomes :

$$x=\frac{100(.00\frac{3}{4}--.01\frac{1}{4})+1\frac{1}{4}(1.0075)^{40}}{.0075\times 1.0075^{40}}=\frac{1\frac{1}{4}\times 1.0075^{40}-\frac{1}{2}}{.0075\times 1.0075^{40}}.$$

$40\log 1.0075=0.0032451\times 40=0.129804$, which corresponds to 1.34835.

$$\frac{1\frac{1}{4}\times 1.34835-\frac{1}{2}}{.0075\times 1.34835}=117.223.$$

$\therefore$ \$$117.22\frac{3}{10}$ is a just price for the bonds mentioned.

Problems of this kind may be solved very readily by arithmetic, as follows :

Take the first problem above. The bond yields \$7. per annum, which is 8% of \$87.50. This would be the price if only \$87.50 were to be paid the investor at maturity. But he will receive \$12.50 more, hence he must now give, in addition to the \$87.50, a sum that will in 20 years at 8% compound semi-annual interest amount to \$12.50.

$$\$12.50\div\$4.80102=2.6036.$$

$\therefore$ \$87.50+\$2.6036=\$90.1036, the price.

When bonds are bought at a premium, the present value must be *deducted* from the sum that would be the price to be paid provided that sum were to be paid the investor at maturity.

Such problems are readily solved, but the arithmetician requires a very complete compound interest table to cover all cases.

The tables used by brokers give the same prices as those obtained by the methods herein set forth ; but they extend only to 6% bonds to run 60 years.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocard's Diameter of a triangle with the circumcircle of this triangle, is either parallel or perpendicular to the besector of the angle formed by the side BC of the triangle ABC and the corresponding side $B'C'$ of Brocard's triangle.

Solution by the PROPOSER.

We shall first prove the following lemma:

1. If upon the sides of the triangle ABC are constructed similar isosceles triangles BA_2C, CB_2A, and AC_2B, and if the perpendicular A_2M_a is produced below BC, so that A'_2M_a is equal to A_2M_a then is $AC_2A'_2B_2$ a parallelogram.

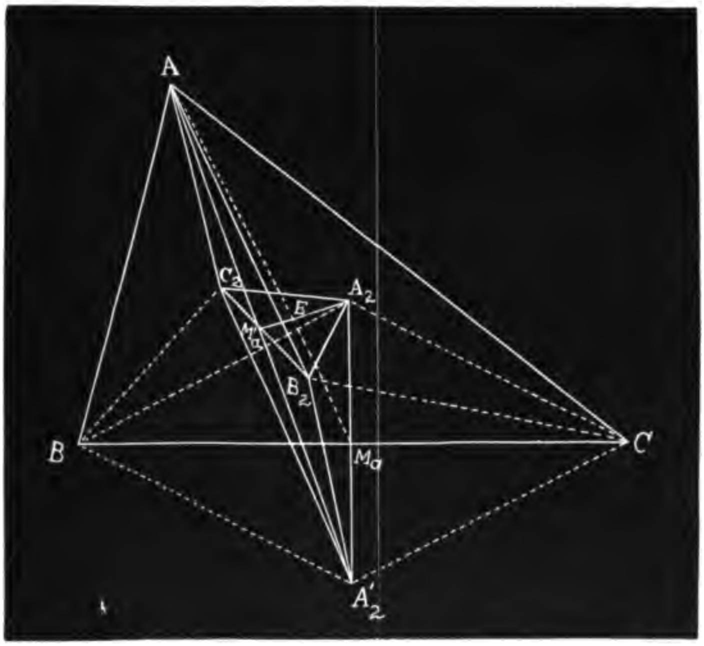


Fig. 1.

$$\sphericalangle C_2BA'_2=\sphericalangle C_2BC+\sphericalangle CBA'_2\ ;$$

but $\qquad \sphericalangle CBA'_2=\sphericalangle A_2BC\ ;$

therefore $\qquad \sphericalangle C_2BA'_2=\sphericalangle C_2BC+\sphericalangle A_2BC\ ;$

but $\qquad \sphericalangle A_2BC=\sphericalangle C_2BA,$

hence $\qquad \sphericalangle C_1BA'_2=\sphericalangle C_2BC+\sphericalangle C_2BA=\sphericalangle ABC.$

The triangle A_2BM_a is similar to triangle C_2BM_c (since they are right triangles having $\sphericalangle A_2BC = \sphericalangle C_2BA$).

Therefore $A_2B : C_2B = BM_a : BM_c = \frac{a}{2} : \frac{c}{2} = a : c$;

but $\qquad A_2B = A'_2B$;

hence $\qquad A'_2B : C_2B = a : c$,

and since the $\sphericalangle A'_2BC_2 = \sphericalangle ABC$, therefore is triangle A'_2BC_2 similar to triangle ABC. In a similar manner can be proved that the triangle $B_2CA'_2$ is also similar to triangle ABC, and therefore A'_2BC_2 and $B_2CA'_2$ are similar to one another. But $A'_2B = A'_2C$ and consequently are the triangles A'_2BC_2 and $B_2CA'_2$ not only similar but also equal and therefore $B_2A'_2 = C_2A$. In a similar manner can be proved that $AB_2 = C_2A'_2$ or $AC_2A'_2B_2$ is a parallelogram.

2. The triangles ABC and $A_2B_2C_2$ have the same median point E.

Since $AC_2A'_2B_2$ is a parallelogram, the diagonals AA'_2 and A_2C_2 will bisect each other at the point M'_a. $A_2M'_a$ is a median line in the triangle $A_2B_2C_2$ as well as in the triangle $AA_2A'_2$. A second median line in the triangle $AA_2A'_2$ is AM_a (since $A_2M_a = A'_2M_a$); we have, therefore, that $A_2E = 2EM'_a$ and $AE = 2EM_a$. But AM_a is also a median line in the triangle ABC, therefore is E the median point in the triangle ABC as well as in the triangle $A_2B_2C_2$.

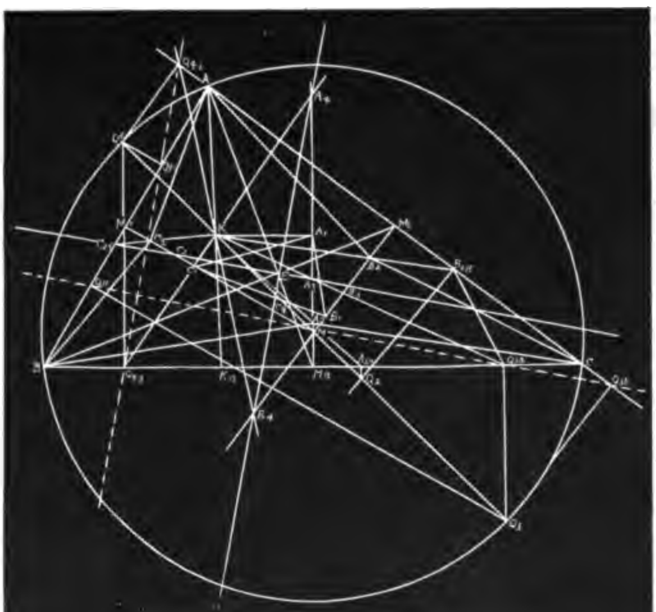


Fig. 2.

A_2, B_2, and C_2 were the vertices of similar isosceles triangles constructed upon the sides of the triangle ABC, and let KA_2, KB_2, and KC_2 meet the sides BC, AC, and AB respectively at $A_{2\alpha}$, $B_{2\beta}$, and $C_{2\gamma}$, then it can be proved that triangle $A_{2\alpha}\ B_{2\beta}\ C_{2\gamma}$ is similar to triangle $A_2\ B_2\ C_2$, the center of similitude being K. If we erect a perpendicular at $A_{2\alpha}$ to BC to meet Brocard's Diameter at Q_2, then, putting for A_1M_a, B_1M_b, their equals KK_a, KK_b respectively, ($A_1B_1C_1$ is Brocard's triangle), we have

$$\frac{A_2M_a}{KK_a}=\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{Q_2M}{Q_2K}.$$

Since the triangles A_2BC and B_2AC are similar, we have

$$\frac{A_2M_a}{B_2M_b}=\frac{M_aC}{M_bC}=\frac{a}{b}=\frac{A_1M_a}{B_1M_b},$$

or
$$\frac{A_2M_a}{A_1M_a}=\frac{B_2M_b}{B_1M_b}=\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K}.$$

Therefore
$$\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{B_{2\beta}B_2}{B_{2\beta}K}.$$

Similarly we get
$$\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{C_{2\gamma}C_2}{C_{2\gamma}K}=\frac{Q_2M}{Q_2K},$$

or, triangles $A_2B_2C_2$ and $A_{2\alpha}B_{2\beta}C_{2\gamma}$ are similar, and K is the center of similitude. From the equation

$$\frac{B_{2\beta}Q_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K},$$

it follows that $B_{2\beta}Q_2$ is parallel to B_2M, and since B_2M is perpendicular to AC, therefore $B_{2\beta}Q_2$ is also perpendicular to AC, or the perpendicular at $B_{2\beta}$ to AC passes through Q_2. Similarly, the perpendicular at $C_{2\gamma}$ to AB passes through Q_2. If, now, Q_2 is made to coincide with either Q_3 or Q_4, the points of intersection of Brocard's Diameter and the circumcircle of the triangle ABC, the triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will then degenerate into the straight lines $Q_{3a}\,Q_{3b}\,Q_{3c}$ and $Q_{4a}\,Q_{4b}\,Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4 with respect to the circumcircle of the triangle. The triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$ and $A_4B_4C_4$, which will be parallel to the Simson lines belonging to Q_3 and Q_4; and they will pass through the median point E, for the lines $A_3B_3C_3$ and $A_4B_4C_4$ still have the median point E in common with ABC.

Also, A_3, B_3, C_3 and A_4, B_4, C_4 are on the perpendiculars at the middle points of the respective sides of the triangle ABC. Since the Simson lines to Q_3 and Q_4 correspond to the extremities of a diameter, they are perpendicular to each other, and therefore their parallels $A_3B_3C_3$ and $A_4B_4C_4$ are also perpendicular to each other.

Furthermore, $Q_{3a}M_a=Q_{4a}M_a$,

$$Q_{3a}M_a : M_aK_a = Q_{4a}M_a : M_aK_a,$$

$$Q_{3a}M_a : M_aK_a = Q_{3a}A_3 : A_3K = A_3M_a : A_3A_1,$$

and
$$Q_{4a}M_a : M_aK_a = Q_{4a}A_4 : A_4K = A_4M_a : A_4A_1,$$

or
$$A_3M_a : A_3A_1 = A_4M_a : A_4A_1,$$

whence $\{M_aA_1,\ A_3A_4\}$ is an harmonic range, and $E\{M_aA_1,\ A_3A_4\}$ is an harmonic pencil. Since $\measuredangle A_4EA_3=90^\circ$, EA_3 will bisect the angle A_1EM_a.

Now, in two similar triangles the bisectors of the angles formed by any line in one triangle with the corresponding line in the other triangle are parallel to each other, hence the bisector of the angle formed by A_1E and EM_a, or the line AE, *i. e.*, the line $A_3B_3C_3$,—which is parallel to the Simson line belonging to one of the points of intersection of Brocard's Diameter, and the circumcircle about the triangle ABC,—is parallel to the bisector of the angle formed by B_1, C_1, and BC. (For particulars I can refer to my Geometrical Treatment of curves which are isogonal conjugate to a straight line with respect to a triangle, published by Leach, Shewell and Sanborn, New York.)

An excellent solution of this problem was also received from Professor G. B. M. Zerr.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

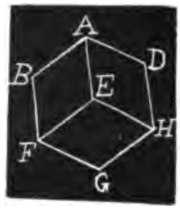
36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsalvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

III. Solution by the PROPOSER.

The $\triangle$ BDE has each side $=\sqrt{2}a$, hence the radius of its circumscribed circle $=\frac{1}{3}\sqrt{6}a$. Hence the distance of A to the plane of $BDE=\frac{1}{3}\sqrt{3}a$. Take the origin at the center of the cube and the line AG as the axis of Z. The revolution will bring each line of the gauche hexagon $EHDCBF$ into either

the position of DH or BF. The equations of DH are $x=\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2z}$, and the equations of BF are $x=-\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2z}$. In either case $x^2=\frac{1}{2}$ and $y^2=2z^2$ and $x^2+y^2=2z^2+\frac{a^2}{2}$ which is the equation of the surface generated by the gauche hexagon $EHDCBF$. This surface could also be generated by the hyperbola $x^2=2z^2+\frac{1}{2}$. Hence the volume of the hyperboloid of one nappe generated $=\int \pi x^2 dz$, the upper limit being $\frac{1}{6}\sqrt{3}a$ and the lower limit $-\frac{1}{6}\sqrt{3}a$. This integral is $\frac{5}{27}\pi\sqrt{3}a^3$.



The lines AB, AE, and AD generate a cone, radius $=\frac{1}{3}\sqrt{6}a$, altitude $=\frac{1}{3}\sqrt{3}a$, volume $=\frac{2}{27}\pi\sqrt{3}a^3$.

The lines GF, GH, and GC generate another cone of the same size.

The sum of the volumes of the three solids $=\frac{1}{3}\pi\sqrt{3}a^3=1.8138a^3$.

[NOTE.—This solution by the Proposer is fuller than that given in the November number, and is published because several of our contributors failed to comprehend the abbreviated solution previously published. Prof. Whitaker asserts that the solution by Dr. Zerr in the September-October number is incorrect, while the latter says he does not as yet see Prof. Whitaker's hyperboloid. The above seems to be correct, but we shall be glad to have the criticisms of other contributors.—EDITOR.]

43. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $\int_2^1 \frac{x^{a-1}+x^{-a}}{1+x}\frac{dx}{x}=\log(\tan\frac{a\pi}{2})$, when $a>0$ and <1.

[*Williamson's Integral Calculus*, p. 15[illegible]]

Comment by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

There seems to be an error in No. 43, as I find the following in my copy of *Williamson*:

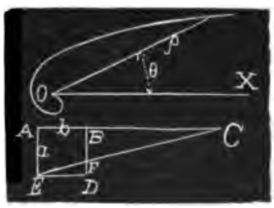
$$\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\frac{dx}{\log x},$$

which gives the required result.

[In *Williamson's Integral Calculus*, edition of 1891, the problem is given as published, but the mistake has doubtless been corrected in the later edition.—EDITOR.]

44. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

Find the equation of a curve in which $\rho=f(\theta)$, in which ρ is equal to BC, an intercept of any secant drawn from the corner E of the rectangle $AEDB$, and prolonged to cut AB prolonged in C. Let equal increments of θ be proportional to the equal increments of DB as divided by the secant EF, θ being zero when EC coincides with ED, and $\theta=2\pi$ when EF passes through B. Determine the asymptotes.



I. Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Referring to the diagram given by the Proposer of this problem, July-Au-

gust (1895) MONTHLY, we have from the similar triangles FBC and FDE the following proportions: $BC:DE::BF:DF$, or $\rho:b::2\pi-\theta:\theta$.

$\therefore\ (\rho+b)\theta=2\pi b$......(1), which is the polar equation of *The Thistle of Scotland*, adopting the suggestion of Prof. MacCord.

Since $\rho^2(d\theta/d\rho)=-[(2\pi-\theta)^2/2\pi]b$, there is a *rectilinear asymptote* parallel to the initial line and at a distance $2\pi b$ *above* it. Making $\theta=\infty$, we have from (1) the equation $\rho=-b$; and this equation characterizes an *asymptotic circle* of radius b, or a *circular asymptote* of same radius, of the curve.

NOTE.—The derivation of (1) can be affected in, at least, *three* different ways; and, according to the conditions of the problem, (1) may also be written

$$(\rho+b)\theta=ab\ldots\ldots(2).$$

II. Solution by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O. Sebastopol, California.

From figure given $\dfrac{BC}{ED}=\dfrac{BF}{DF}=\dfrac{BD-DF}{DF}=\dfrac{BD}{DF}-1$,

or $\dfrac{\rho}{b}=\dfrac{2\pi}{\theta}-1$; $\rho=\dfrac{2b\pi}{\theta}-b$, the equation of the curve.

When $\theta=0$, $\rho=\infty$, and subtangent $=-2b\pi$.

The curve has, therefore, an asymptote parallel to OX at a distance above it, $2b\pi$, the circumference of a circle with radius AB.

The curve is concave toward the pole and intersects the axis perpendicularly and at a distance b to the left of the pole.

Elaborately solved by *O. W. ANTHONY*, and *C. W. M. BLACK*.

ERRATA.—On page 363, of last issue, line 4. omit $\sqrt{3}$ in the numerator of the second term; line 9, in the numerator, for "(a^2+x^2)" read (a^2-x^2); line 11, in the denominator of the second term, for "4" read 4^2; line 14, for "+" read $=$, before the last expression; page 364, line 15, for "of" read *to*; line 17, insert comma after "length"; line 17, for "$2n$" read 2π; line 18, for "π_2" read π^2; on same page, problem No. "43" should be No. 42; page 365, line 1, for "z^{n-2}" read z^{n-1}; and in line 2, of solution III., for "n^2+y^2" in the exponent, read x^2+y^2.

PROBLEMS.

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51. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustrum.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area.

EDITORIALS.

In this issue will be found a bill with the amount due us to the end of 1896 marked thereon.

The great work of preparing the list of contributors and the index for Vol. II. is to be credited to Editor Colaw.

Prof. G. H. Harvill is now permanently located at Athens, Texas, from which place the *Mathematical Messenger* will be issued.

Persons wishing to discontinue their subscriptions to the MONTHLY, and who are not in arrears, should return this number with their names written upon the wrapper.

Mr. John McDowell of Philadelphia, writes as follows: "Find enclosed three dollars, being amount of subscription for your valuable journal, THE AMERICAN MATHEMATICAL MONTHLY, for '96."

This number of the MONTHLY has been cut short in order that we may catch up in its publication. We shall cut the February number some also. The March number will contain the regular departments again.

Prof. P. S. Berg, Larimore, North Dakota, writes, "Enclosed find three dollars as my subscription to THE AMERICAN MATHEMATICAL MONTHLY. . . . I should not be without it if the subscription price were five dollars."

Dr. G. A. Miller, Leipzig, Germany, writing in reference to the MONTHLY, says, "When I return I hope to be able to do much more towards aiding such efforts towards advancing the cause of mathematics in the United States. You are doing a great work. I hope you will not be discouraged in it."

Our valued contributor, Dr. Alexander Macfarlane, has an article on Quaternions in *Science* of January 17th. He has also prepared the article on Vector Analysis and Quaternions in *Higher Mathematics for Engineering Colleges*, a work edited by Drs. Mansfield Merriman and Robert S. Woodward, and which is expected to be ready in July.

Dr. G. B. M. Zerr, of Texarkana College, says, in a letter of January 7th, "I will remit subscription for '96 in a few weeks. I will remit $3.00 and am willing to pay $5.00 if necessary. I find myself very much benefited by the excellent solutions and excellent papers that appear in each number of the MONTHLY. Do not allow its publication to cease, rather raise the subscription price, I am satisfied the subscribers will stand by you."

NOTES.

Drs. Fisher and Schwatt's translation of Dr. H. Durège's Elements of the Theory of Functions is now ready.

Alexander Macmillan, the younger of the two brothers of the firm Macmillan & Co., died in England on January 25.

THE LOBACHEVSKI PRIZE.

On May 1, 1895, the Lobachévski Fund had reached, beyond all expenses, 8840 roubles, 95 kopeks.

This sum permits the accomplishment of the double aim of the committee: to found an international prize for research in geometry, especially non-Euclidean geometry, and to erect a bust of the celebrated scientist.

The prize, 500 roubles, will be adjudged every three years to the best works or memoirs on geometry, especially non-Euclidean geometry.

The prize will be given for works printed in the Russian, French, German, English, Italian, or Latin, sent to the Physico-Mathematical Society of Kazán by the authors, published during the six years which precede the adjudication of the prize. Works to compete must be sent to the Society at the latest one year before the day of award, October 22 old style (November 3).

The first prize will be adjudged October 22 (November 3), 1897.

To award the prize, the Society will form a commission to choose judges among Russian or foreign scientists.

The work of the judges (reporters) will be recompensed by medals of gold, bearing the name of Lobachévski.

As a fixed capital to found this prize, 6000 roubles were invested.

Of the sum collected, an additional 2000 roubles goes to share the expense of erecting a bust of Lobachévski in the park bearing his name in front of the University edifice in Kazán, the remainder of the cost to be borne by the Municipal Council.

A special committee, consisting of representatives of the Municipal Council and of the Physico-Mathematical Society, has made a contract with Mlle. Dillon, who engages for 3000 roubles to furnish a bronze bust of Lobachévski, to be placed on a granite pedestal, the height of the monument to exceed 3 mèters.

It is hoped to unveil the bust between the 15th and 25th of September, 1896.

This 'fête mathématique' will follow the 'congrès des savants russes naturalistes et mathématiciens' at Kiev from 1st to 12th of September, 1896, and be during the grand Russian Exposition artistic and industrial at Nijny-Novgorod in the summer and autumn of 1896. Foreigners in any way identified with the name of Lobachévski are invited to the fête, and such as accept will be the guests of the city and University of Kazán.

For a second bust of Lobachévski to be placed in the Assembly Hall of the

University, 200 roubles have been given from the Lobachévski fund, the remainder of the cost to be borne by the professors of the University.

The residue of the sum already collected (640 r. 95 k.) will be added to the fixed capital. The augmentation of the capital will permit of a new edition of Lobachévski's works in a few years, the first volume of the Kazán edition having already become rare (out of print).

The Physico-Mathematical Society of Kazán has already received a large number of works and memoirs relating to Lobachévski and non-Euclidean geometry, and now having added its own collection of the printed and manuscript works of Lobachévski, the Society has inaugurated a separate library under the name *Bibliotheca Lobachévskiana.* It is hoped that in time this library will collect all the literature of non-Euclidean geometry and be an indispensable aid to those engaged in its development.

All writers on this fecund subject are begged to send to this library copies of their works.

Alas! That the Mathematico-physical Society of Hungary, a country having an equal claim to all the honors of the non-Euclidean geometry through the genius of Bolyai János, should have been content with placing in 1894 a monumental stone on his long neglected grave in Maros-Vásárhely!

GEORGE BRUCE HALSTED.

Austin, Texas.

THE UNIVERSITY OF CHICAGO: SUMMER, 1896.

The following mathematical courses will be offered: By Professor *Moore*, Theory of numbers, Differential equations (with introduction to Lie's continuous transformation groups); by Professor *Bolza*, Theory of substitutions, Theory of functions of a complex variable; by Professor *Miller*, of the University of Indiana, Analytical geometry of three dimensions; by Dr. *Young*, Conferences on mathematical pedagogy, Theory of equations, College algebra; by Mr. *Slaught*, Advanced integral calculus, Introductory course in differential and integral calculus; and by Mr. *Baker*, Analytical geometry of the plane. The pedagogical conferences are two hours weekly for six weeks and the other courses are four or five hours weekly for twelve weeks from July 1, 1896. Those who expect to work in mathematics in the University of Chicago during the coming summer as well as those who desire further information are requested to communicate with Professor Moore.

BOOKS AND PERIODICALS.

Elementary Mensuration. By F. H. Stevens, M. A., Formerly Scholar of Queen's College, Oxford; A Master of the Military Side, Clifton College. 12mo. cloth, 243 pp. Price, 90 cents, net. New York: Macmillan & Co.

This text-book of Elementary Mensuration is divided into two parts. The first part provides for those students whose knowledge of Geometry is confined to Euclid's First Book, and Algebra to the meaning of the simplest symbols. In the second part more difficult questions are offered to students who have mastered the Sixth Book of Euclid, have attained some facility in ordinary Algebraical methods as far as the Binomial Theorem and have made a beginning with Trigonometry.

Under each rule is given an illustrative solution neatly worked out, and proofs of formulæ have been given or indicated whenever they seemed likely to be intelligent to the learner. The book is in every way worthy of the consideration of teachers who are needing a good elementary text on Mensuration. B. F. F.

Problems in Differential Calculus Supplementary to a Treatise on Differential Calculus. By W. E. Byerly, Ph. D., Professor of Mathematics in Harvard University. 8vo. cloth. viii and 72 pp. Price, 80 cents. Boston and Chicago: Ginn & Co.

An excellent collection of about 350 problems to supplement the author's Treatise on the Differential Calculus. While these problems were especially prepared to use in connection with Dr. Byerly's Calculus they will be found useful wherever the subject is studied. B. F. F.

Computation Rules and Logarithms with Tables for other Useful Functions. By Silas W. Holman, Professor of Physics at the Massachusetts Institute of Technology. 8vo. cloth, 73 pp. Price, $1.00, net. New York: Macmillan & Co.

Besides a Table of Five Place Logarithms containing an abbreviated Table for One and Two Place Numbers, a table for five place numbers from 1.0 to 1.1, avoiding interpolation, a table for all four place numbers with interpolation tables for the fifth place; a table of logarithms of sines, cosines, tangents, and cotangents to four places; and a table of logarithms of sines, cosines, tangents. and cotangents to five places; there is also a four place logarithm table of numbers from 1 to 10; a table of square roots and squares of numbers from 1 to 100; a table of reciprocals of numbers from 1 to 1000; a table of slide wire ratios; a table of natural sines, cosines, tangents, and cotangents, and a number of tables of mathematical constants.

A very useful book for the practical computor. B. F. F.

Algebra for Schools and Colleges. By William Freeland, A. B., Head Master of the Harvard School, New York City. 8vo. cloth, 310 pp. Introduction Price, $1.12. New York: Longmans, Green & Co.

With the exception of two or three instances, the author sets no claim to originality. The book is designed to meet the requirements of those students who present themselves for the maximum courses in Freshman work for students who have advanced through the subject of Quadratics only.

Throughout the course tests for revision have been inserted, and a collection of 500 carefully graded Miscellaneous Examples has been given at the end of the book. The number of examples in the book is 5,200. It is very neatly printed on a good quality of paper. B. F. F.

A Primer of the History of Mathematics. By W. W. Rouse Ball, Fellow and Tutor of Trinity College, Cambridge, England. 12mo. cloth, 162 pp. Price, 65 cents. New York: Macmillan & Co.

This most charming little book ought to be used in all Algebra and Geometry classes in order to awaken early an interest in the History of Mathematics. A few years ago, I gave a short lecture to a class of about 60 students in Algebra, on the Arabic System of Notation. After the lecture, a young man said to me, "Is it possible that Arithmetic and Algebra have come down to us in their present form by a gradual development. I thought they were always as they are now." Were some such work as Mr. Ball's Primer used in our classes in Algebra and Geometry, such dense ignorance concerning one of the greatest departments of human knowledge would not exist. No one having then studied Arthemetic would suppose that the subject sprung from the human mind as perfect as Minerva from the head of Jupiter. B. F. F.

The Elements of Physics. A College Text Book. By Edward L. Nichols and William S. Franklin. In three volumes. Vol. I. Mechanics and Heat. 8vo. cloth, 228 pp. Price, $1.50. New York: Macmillan & Co.

In this valuable treatise on Physics, the authors have not attempted to lift the student over difficulties and set him down in easy places. The work, it appears, is written with a view of giving the student the best possible advantage of the subject. The authors have squarely faced the difficulties of the subject and have, as occasion demanded, used the Calculus rather than encounter a subject by long, laborious and indirect methods avoiding the use of the Calculus. However, the degree of mathematical experience of the undergraduate reader has been kept in view and the various proofs and demonstrations have been given the simplest possible form. The concepts of directed and distributed quantity are briefly treated in Chapter II of Vol. I.

From what we know of the first volume we believe that this Treatise will prove to be the best that has yet appeared in this country. B. F. F.

The Basis. A Monthly Magazine. Devoted to Good Citizenship. Edited by Judge Albion W. Tourgee, Mayville, New York. Price, $1.50 per year.

The Basis for January is a pleasant surprise in its new cover. The leading editorial denounces the retirement of the greenback as an "Epoch-Making Crime." In "A Bystander's Notes", Judge Tourgee treats especially the lack of earnest effort on the part of the colored race for the betterment of their condition. The Mob-Record, the Department of Good Government Clubs and "Today's Thought" are well in evidence. There is a good short story and other characteristic matter. The number speaks well of the new management of *The Basis* and its new home on the Chautauqua Hills.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

The *Review of Reviews* for February contains an article which, in the compass of two pages, makes perhaps the most telling and effective exposure of the recent Turkish massacres that has yet been attempted in the English language. The article is based upon full accounts of the massacres, written on the ground by trustworty and intelligent persons—French, English, American, Turk, Kurd, and Armenian—who were eye-witnesses of the terrible scenes. The article estimates the number of killed in the massacres thus far at 50,000, the property destroyed at $40,000,000, and the number of starving survivors at 350,000.

Elements of the Theory of Functions of a Complex Variable with especial reference to the methods of Riemann. By Dr. H. Durege, late Professor in the University of Prague. Authorized translation from the fourth German Edition. By George Egbert Fisher, M. A., Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania, and Isaac J. Schwatt, Ph. D., Instructor in Mathematics in the University of Pennsylvania. Large 8vo. cloth, 288 pp. Price, $2.50. Philadelphia: G. E. Fisher and I. J. Schwatt.

This valuable work comes to us just in time for notice in this issue of the MONTHLY. From only a cursory examination of it, we do not hesitate to emphasize what was said of it in the last issue. The work will afford a most excellent introduction to the study of the Theory of Functions and the intelligent reading of the larger Treatises—such as Forsyth's.

The mechanical and typographical execution of the book is first class. B. F. F.

The Number Concept. Its Origin and Development. By Levi Leonard Connant, Ph. D., Associate Professor of Mathematics in the Worcester Polytechnic Institute. 8vo. cloth, 218 pp. Price, $2.00. New York: Macmillan & Co.

This work forms a most valuable addition to the literature of mathematics. The first chapter treats on Counting; the second, Number System Limits; the third and fourth, Origin of Number Words; the fifth, Miscellaneous Number Bases; the sixth, The Quinary System; the seventh, The Vigesimal System.

The treatment of these subjects is very interesting and evince careful study and research. B. F. F.

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BIOGRAPHY.

EMILE-MICHEL-HYACINTHE LEMOINE.

BY DAVID EUGENE SMITH, PH. D., PROFESSOR OF MATHEMATICS IN MICHIGAN STATE NORMAL SCHOOL, YPSILANTI, MICHIGAN.

SO EXTENSIVE has become the modern geometry of the triangle that one scarcely realizes that it has almost entirely developed within the last quarter of a century, and that most of its discoverers are still among the living. Lemoine, Brocard, Neuberg, Tucker, and W. J. C. Miller whose mathematical work in the *Educational Times* has done so much for the subject,—these and many others have lived to see their labors crowned with honor by lovers of geometry.

To none of these more than to Emile-Michel-Hyacinthe Lemoine is due the honor of having started this movement, and to him is the following brief sketch devoted.

M. Lemoine was born at Quimper, Finistère, in the west of France, Nov. 22, 1840. His father, a retired captain, who had been in all of the campaigns of the Empire after 1807, placed him as foundation scholar in the military Prytanée of La Flèche, whence he proceeded to the École Polytechnique. He entered this great breeding place of mathematicians at the age of twenty, the year of his father's death, and completed the course in due time. Instead of accepting any of the careers offered by the State to all graduates of the Polytechnic School, M. Lemoine determined to make his own way. Indeed, for the next few years, although engaged in science teaching in Paris, he seems to have run the round of pleasure of which that city is the home *par excellence.* Of great versatility and exceptional conversational powers, with an originality that fascinated and a per-

sonality that impressed his large circle of friends, he lived the life of a *dilettante* in the best sense of the term, and drank at the fountains of pleasure, of politics, of the arts, and of the sciences.

In these days Lemoine led as varied a life in education as in the less scholastic walks. We find him a student in the École des Mines, then *preparateur* of M. Janssen at the École d'Architecture,—supplying the place of his former professor, M. Kiœs, in the preparatory course of the École des Beaux Arts,—perfecting his knowledge of chemistry in the laboratory of Wurtz, for whom he always had a great admiration and between whom and himself there was much affection,—frequenting the courses of the École de Médecine, the hospitals and the clinics,—dabbling in philology,—and ending up by trying the law for a year. This last fancy he was forced to forego because he found himself in disgrace with the Empire through his republican principles and his liberal views on church matters. During these years, too, Lemoine traveled as his income would allow, and when his income failed him he not infrequently traveled as tutor in some wealthy family. Thus it was that he started out in his work as a teacher, full of life and health and hopes, although possibly scattering his attention too much for a career of highest success.

But however the result may have been, an unforeseen accident nipped the experiment in the bud. In 1870, when only a little more than twenty-nine years old, a laryngeal difficulty put an end to his teaching, and required him to leave Paris and seek rest at Grenoble. In the army for a time, he returned to Paris a couple of months after the Commune, and for a number of years filled divers positions in the engineering line. Finally, in 1886, he was appointed city engineer at the head of the gas department, a position which he still holds.

It is, however, with his mathematical work that we are concerned directly. In 1871 he, together with eight or ten other mathematicians, issued the circular which started the Société Mathématique of France. He was among the first to follow and to assist d'Almeida in founding the Journal de Physique and the Société de Physique. He joined with Wurtz, Friedel and others in the organization of the Association Française pour l'Avancement des Sciences. It was while yet a boy in his teens at La Flèche, that, in 1858, he published a short note in the Nouvelles Annales de Mathématiques, which discussed certain properties of the triangle. But it was at the Congrès de Lyon of the Association Française pour l'Avancement des Sciences, in 1873, that he presented his brief but noteworthy paper *Sur quelques propriétés d'un point remarquable de triangle*, and thus, as Casey says, made himself known as the founder of the modern geometry of the triangle. In the same year he published a short note in the Nouvelles Annales on the same subject. In 1874 he presented at the Congrès de Lille a second paper on the geometry of the triangle, entitled *Note sur les propriétés du center des médiances antiparalleles dans un triangle*, a point which has since been quite generally known as the *Lemoine point*, althoueh it is also called the *symmedian point* in England, and the *Grebe point* in Germany. The first paper (1873) contains among others the familiar theorem which may now be

stated thus: "The three parallels to the sides of a triangle through its Lemoine point meet the sides in six concyclic points (the first Lemoine circle)." By the Lemoine (symmedian) point is meant the point of concurrence of the symmedians of a triangle. Since the appearance of these two papers, Lemoine's name has been familiar to all readers of the mathematical journals in every country, and it is for these contributions that he seems destined to be known, rather than for his *Géométrographie* which he considers his greatest work.

La Géométrographie, of which he had the first ideas in 1888, was suggested by him in a memoir, on a more general theme, presented to the Congrès d' Oran of the Association Française pour l'Avancement des Sciences. The title of the paper is *De la mesure de la simplicité dans less Sciences mathématiques*, but for lack of time the study was limited to the simplicity of geometric constructions. On the same subject he published a short note in the *Comptes Rendus* of the Academy for that year,—more strictly *Sur la mesure de la simplicité dans les constructions géométriques.* Since then he has published numerous articles on the same or kindred subjects, in various journals, among them Mathesis (1888), Journal des mathématiques élémentaires (1889), Nouvelles Annales de Mathématiques (1892), in which last named article he considers especially the Problem of Apollonius. Finally, in 1892, at the Congrès de Pau and again at Besançon in 1893 and at Caen in 1894, a series of papers was presented on *La Géométrographie ou l'art des constructions Géométriques*, which may be considered as closing the subject of "geometrography" as applied either to the geometry of the rule and compasses alone, or to those constructions which admit the square, as in descriptive geometry.

Next in importance to the subject of "geometrography," M. Lemoine ranks his work on Continuous Transformation which permits of forming without effort, almost mechanically, a great number of formulæ and theorems relative to the triangle and to the tetrahedron. The principal memoirs which he has presented on this subject are the following: *Sur les transformations systématiques des formules relatives au triangle*, Congrès de Marseille 1891; *Étude sur une nouvelle transformation dite transformation continue*, in Mathesis for 1891; *Une règle d'analogies dans le triangle et la specification de certaines analogies à une transformation dite transformation continue*, in the Nouvelles Annales for 1893; and finally a memoir entitled *Applications au tétraèdre de la transformation continue.*

Three other geometric studies have been undertaken by M. Lemoine, which deserve especial mention. One is the study of *Triangles Orthologiques.* Steiner demonstrated that if two triangles ABC, $A'B'C'$ are such that the perpendiculars drawn from A, B, C, respectively, on $B'C'$, $C'A'$, $A'B'$ are concurrent, then, reciprocally, the perpendiculars drawn from A', B', C', on BC, CA, AB, respectively, are concurrent. Lemoine calls these triangles *orthologiques* and makes them the basis of a theory developed in several memoirs, notably in one presented at the Congrès de Limoges in 1890. He has also published three papers on the application of geometry to the calculus of probabilities, in the Bulletin de la Société Mathématique (1883), the Nouvelles Annales (1884), and

the proceedings of the Congrès de Grenoble (1885). And finally, there should be mentioned a memoir presented at the Congrès de Nantes in 1875, entitled *Étude systématique du tétraèdre equifacial* (in which the four faces have equal area.)

But in some respects the crowning labor of M. Lemoine is the creation of *L'Intermédiaire des Mathématiciens*, the details of which should be told as a matter of historic interest, especially as they have not heretofore appeared. This publication, although still in its infancy, is known throughout the mathematical world. It consists simply of questions and answers, questions which one asks for information and not for the mere pleasure of displaying some puzzle, questions which bring one into a kind of personal relation to his co-workers whether they be in Russia or South Africa. The idea of the journal is purely M. Lemoine's, and for some time it had been in his mind, but unhappily with no thought of its realization, until the genial influence of a quiet dinner and some good cigars brought about its fruition. M. Laisant had long been a friend of Lemoine's, and it was no uncommon thing for the former to dine with the latter at his home in Rue Littré. On such an occasion, in March, 1893, as they were enjoying a quiet smoke after dinner, the talk ran as usual into mathematics, and Lemoine suggested the idea of the journal. Laisant at once saw the value of the scheme and urged his friend to join him in carrying it out. M. Lemoine replied that it seemed impossible both because he was much occupied with other matters, and because of ill health (from which, unhappily, he is still suffering). Nevertheless, M. Laisant was so persuasive and the influence of the dinner and the cigars so happy that before they separated the project had taken such form that the very next day it was laid before their friend Gauthier-Villars, the great mathematical publisher, and the journal was ushered into being. "Before dinner, nothing could have persuaded me," M. Lemoine writes, "that this idea which I had formed for others would ever be realized by me ; after dinner, the journal was a possibility ; the next day, it was an accomplished fact." Its publication began in January, 1894, and each editor serves during six months of the year.

As one surveys the labors of Lemoine it would seem, from present appearances, that his most valuable work is the foundation of *L'Intermédiaire*, a publication which bids fair to continue for generations because it is really needed. His most original mathematical work seems to be his "geometrography,"—purely a creation of his own, and a contribution which enters into the mathematical work of the military schools of Brussels and Turin, the polytechnic schools of Zurich and Milan, and more or less in many other places. The work which will bring his name to the most readers is his study of the modern geometry of the triangle. In general it may be said that his contribution to geometry has been the very valuable work of showing that the synthetic field is by no means exhausted ; that Euclid left something for this generation to accomplish ; and that an original mind can find abundant material in even so simple a figure as the simplest polygon. How suggestive is this of the vast field which awaits investigators of the more complex geometric figures !

This sketch should not close without a brief reference to the influence that M. Lemoine has exerted in the realm of music. The soirées of M. and Mme. Lemoine are justly celebrated, and each week of the winter sees an assemblage representing the *anciens élevées* of the École Polytechnique, the École Normale, the Marine, and in general a good part of the scientific, literary, and artistic circles of Paris, to listen to a musical programme as original as the mathematical labors of the host. These soirées have exerted a great influence in a musical way, the type which they have fixed being adopted by many societies in and about Paris. One amusing feature of these meetings is the name which designates them. If the writer may be pardoned a personal allusion, he once attended an examination in the École Polytechnique by M. Hermann Laurent. It was one of the most severe he had ever seen,—an exceptionally bright young man submitted to an oral examination that would certainly have floored most American professors,—the examiner, a dyspeptic looking man as cold and as keen as steel and apparently as unsympathetic as ice, though in reality one of the most genial of men. To this justly celebrated mathematician, M. Laurent, is due the name of M. Lemoine's soirées, "La Trompette." Long ago he one day remarked to M. Lemoine in a jesting way, as the latter was excusing himself to attend one of his musical reunions, "Stay here with me, let the trumpet alone." Struck by the name, Lemoine adopted it, and *La Trompette* has ever since designated the delightful soirées with which the Paris cultured world is familiar.

A final word concerning the modesty of M. Lemoine. He estimates his position exactly. He says that he is not a mathematician. He has no claim to rank with Hermite, Poincaré, Picard, Painlevé, Appell, Jordan, Bertrand, Tannery, Darboux, or any of that famous circle which is making Paris such a center of study in the fields of higher modern mathematics. But all mathematicians feel that he has done a noteworthy work in other lines, and for this his name will be known and prominently known in the history of mathematics.

Ypsilanti, Michigan, March, 1896.

WHERE MATHEMATICIANS ARE NEEDED.

By ERIC DOOLITTLE, A. M., Chicago, Illinois.

There is no study of which the conceptions are more grand, nor of which the theorems are more comprehensive and profound than the study of Physical Astronomy. There is no study affording an application of Pure Mathematics in which the perfect harmony of its various parts is more evident; none in which

reason plays a greater part nor approximation a less one. The beauty and simplicity of its first propositions richly reward the early attention of the student, and in the end he is led to the wonderful theorems of *La Place* on the stability of the solar system and the conditions of its formation; theorems which *Baron Fourier* has justly named the highest which the human intelligence can propose.

It is remarkable that more young mathematicians do not enter this absorbing field. The common impression that it requires an unusual mathematical training is largely erroneous. Such a thorough knowledge of Calculus and Mechanics as is shown by the many contributors of the MONTHLY is fully sufficient. Physical Astronomy demands patient and steadfast work; mere brilliance and versatility can accomplish no more of fundamental importance here than in any other true science.

I would urge upon those who are now fitted to enter this or other like work, the great necessity of concentrating their energies upon it. It should be the one object of every devoted student to perfect and advance his own science. It is to this that his whole work must be directed. To such an one years of fragmentary study, first on one subject and then on another, are utterly wasted.

It is the disastrous mistake of many students that they do not realize how soon study for mere amusement or culture should give place to something higher. They fear, often mistakenly, that they are incapable of beginning work of real importance: instead of arranging then a definite series of studies to prepare themselves, they continue to dissipate their strength and accomplish nothing.

Physical Astronomy is calling in many directions for original work. In this country it is comparatively neglected. There are many who are being attracted by the pleasures of Photography and Spectroscopy, but there are few who realize the field which the Fundamental Astronomy opens to them. It contains many problems of the deepest interest. It is filled with questions whose answer requires, not an expensive observatory, but rather mathematical patience or skill.

Readers of the MONTHLY who are determined to accomplish something may well devote themselves to this science. The certainty of their adding to the sum of human knowledge is here greater than in Pure Mathematics, the reward of faithful work unaccompanied by special genius far more certain. The explanation of the variable stars, of the cause and nature of the sun's peculiar rotation, more complete theories of the satellites and of the figures and attractions of the Heavenly Bodies, the determination of the perturbations of the asteroids and other planets and the causes of the anomalies which occur, and the able discussion of a multitude of observations relating to these and other problems are a very few of the many directions in which original work is needed.

As with any true science, Physical Astronomy requires from those who enter upon it long and patient devotion. Its rewards are not bestowed by

chance, nor are they on that account of less value. It is of little popular interest. Its discoveries are seldom sensational. But its dignity and importance cannot be over-estimated. Of American Astronomers, the names of Hill and Newcomb will go down through the ages: their researches will never lose their importance. And whoever adds to this science is contributing to a knowledge which shall endure forever.

Baron Fourier said of La Place:

"Your successors, gentlemen, will witness the accomplishment of the great phenomena whose laws he discovered. They will observe in the motions of the Moon the changes which he predicted and of which he alone was able to assign the cause. The continued observation of Jupiter's satellites will perpetuate the memory of the inventor of the laws which govern them in their courses. The great inequality of Jupiter and Saturn, running through their long periods, and giving to these bodies new situations will recall without ceasing one of his most astonishing discoveries. These are the titles of a true glory which nothing can extinguish. The spectacle of the heavens will be changed, but at those remote epochs the glory of the inventor will continue forever; the traces of his genius bear the seal of immortality."

Chicago University, February 9th, 1896.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXII. *If two straights AB, CD existing in the same plane stand perpendicular to a certain straight BD; but AC joining these perpendiculars makes with them internal acute angles (in hypothesis of acute angle): I say (Fig. 26) the terminated straights AC, BD have a common perpendicular, and indeed within the limits fixed by the designated points A and C.*

Fig. 26.

Proof. For if AB, CD are equal, it follows (from P. II) that the straight LK, by which these two AC and BD are bisected, will be to them a common perpendicular. But if either be the greater, as suppose AB; let fall to BD (according to Eu. I. 12) from any point L of AC the perpendicular LK, meet-

ing the other BD in K. But it will meet it in some point K existing between the points B and D; otherwise (contrary to Eu. I. 17) the perpendicular LK would cut either AB, or CD, perpendicular to the same BD. So if the angles at the point L are not right, one of them will be acute and the other obtuse. Let the obtuse be toward the point C. But now LK is understood so to proceed toward AB, that it always stands at right angles to BD, and likewise opportunely increased, or diminished, in some point of it cuts the straight AC. It follows that the angles at the intersection points with AC cannot all be obtuse toward the parts of the point C, lest at length in that point A, where the straight LK is congruent with the straight AB, the angle at the point A toward the parts of the point C should be obtuse, when toward these parts it is by hypothesis acute. Since therefore the angle at the point L of this LK is by hypothesis obtuse toward the parts of the point C, the straight LK will not change over in this motion so as to make in some point of it with the straight AC an angle acute toward the parts of the aforesaid point C, unless, before, it changes over so as to make in some point of it with this AC an angle right towards the parts of this same point C. Therefore between the points A, and L will be some one intermediate point H, in which HK perpendicular to this BD is also perpendicular to to the other AC.

In a similar manner is shown to be present a certain XK between LK, CD, which is perpendicular both to the straight BD, and to the straight AC, if namely an angle at the point L is assumed to be obtuse toward the parts of the point A.

It follows therefore that the strights AC, BD will have a common perpendicular, and indeed within the limits fixed by the designated points A, and C, when the joins AB, CD exist in the same plane and are perpendicular to BD.

Quod erat, etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from January Number.]

(2) For each generating substitution s_α in a transitive group of degree m_1 find a substitution s_β which (a) interchanges the systems in the same way as s_α interchanges its elements, (b) has its k^{th} power in G_1 where k is the *order* of s_α, *i. e.* the lowest positive value of x which satisfies the equation $s_\alpha^x=1$, and (c) if s_β is

the first substitution which corresponds to a generating substitution in the group of degree m_1, s_β needs only to transform G_1 into itself; otherwise s_β must transform the group already found in the same way as s_a transforms the corresponding part of its group. Continue until all the generating substitutions s_a have been used. We will thus obtain a non-primitive group.

(3) Determine whether the non-primitive group just found is different from each one of those already in the list.

The relation which exists between the required non-primitive group G and the given group of degree m_1 G^1 is called a g_1, 1 *isomorphism*, or a g_1, 1 *correspondence*. The problem of constructing all the non-primitive groups of degree n has its more difficult elements in common with the problem of establishing an a, b correspondence between two groups as may at once be inferred from the given relation. We shall not pursue this subject for the present since only its most evident principles need to be employed when n is small.

To this development of the elementary methods pursued in the construction of non-primitive groups we will add a proof of the general theorem to which we referred in a foot-note. For the sake of simplicity we shall not give the theorem in its most general form.

Theorem. Given that the number of the systems of non-primitivity is n and that the group which does not interchange the systems G_1 is the product of n conjugate transitive groups of which one is found in each system, then there is only one non-primitive group based upon the given G_1 and isomorphic to a transitive group of n elements which is generated by a single substitution.

There certainly is one such group for we may chose s_β so that it will simply permute the systems in the same way as s_a permutes its elements and will have the same order as s_a. Since s_β simply permutes the systems, *i. e.* it permutes the systems as units without permuting the elements of the systems, it must also transform G_1 into itself. Hence G_1 and s_β generate a non-primitive group whenever G_1 differs from identity.

Let t_1^1, t_2^2, ... t_n^n(the upper index standing for the systems in the same order as they are represented by the letters of s_a and the lower index for the particular substitution in the system) be any substitution in the n systems which transform the n constituents of G_1 into themselves. Then will

$$t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta$$

be a symbol for all the substitutions whose degree $\overline{\overline{<}}$ the degree of the required group which transform G_1 into itself and permute the systems in the same way as s_a permutes its elements. If this general substitution satisfies the other condition which must be satisfied if it, with the given G_1, generates a non-primitive group we have

$$(t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta\,)^K = \text{some substitution in } G_1,$$

where K is the smallest positive value of x in the equation

$$s_a^x=1.$$

Since $(t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta)^K = t^1_1\, t^1_2 \ldots\ldots t^1_n\, t^2_2\, t^2_3 \ldots\ldots t^2_1 \ldots\ldots t^n_n t^n_1 \ldots\ldots t^n_{n-1}$

we know that $t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta$ may be multiplied by some substitution of G_1 so as to give for the new t's

$$t^1_1\, t^2_2 \ldots\ldots t^n_n = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Consider now the equations

$$(K^1_1\, K^2_2 \ldots\ldots K^n_n)^{-1} s_\beta\, K^1_1\, K^2_2 \ldots\ldots K^{n*}_n =$$

$$\overset{-1}{K}{}^n_n K^n_1\, \overset{-1}{K}{}^{n-1}_{n-1} K^{n-1}_n \ldots\ldots \overset{-1}{K}{}^1_1\, K^1_2\, s_\beta = t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta\,.$$

We see directly that the following is a solution of the last equation if (A) is satisfied:

$$K^1_1=1,\ K^2_2=t^2_1,\ K^3_3=t^3_1\, t^3_2,\ldots\ldots K^n_n=t^n_1\, t^n_2 \ldots\ldots t^n_{n-1}.$$

Hence all the possible groups are conjugate to the one already given and our theorem is proved. This theorem may be employed with respect to the first subgroups as well as with respect to the entire groups.

In our next paper we shall consider the construction of the third and last class of groups, viz: the *primitive* groups.

[To be Continued.]

ON AN INTERESTING SYSTEM OF QUADRATIC EQUATIONS.

By DR. E. H. MOORE, University of Chicago, and EMMA C. ACKERMANN, Michigan State Normal School.

In C. Smith's Algebra, fourth edition, p. 134, are given for solution, examples 61, 62, 63, which are as follows (the third with a slight modification):

61. The roots of the equation $x^2+mx+m^2+a=0$ are $x_1,\ x_2$; show that $x_1{}^2+x_1x_2+x_2{}^2+a=0$.

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*This equation follows from the simpler one

$$(ts)^{-1}=s^{-1}t^{-1}$$

and this is true because if we multiply both members by ts we obtain an identity.

62. The roots of the equation $(x^2+1)(a^2+1)-max(ax-1)=0$ are x_1, x_2; show that $(x_1{}^2+1)(x_2{}^2+1)-mx_1x_2(x_1x_2-1)=0$.

63. The roots of the equation $a(x^2+mx+m^2)+bm^2x^2=0$ are x_1, x_2; show that $a(x_1{}^2+x_1x_2+x_2{}^2)+bx_1{}^2x_2{}^2=0$.

The equations possess the following properties: (1), the equation is of the second degree in the variable x and the constant a; (2), the roots x_1, x_2 of the equation are related to each other exactly as are the variable x and constant a.

We seek to generalize these theorems and formulate this problem:

To determine all quadratic equations of the form

$$f(\overset{2}{\smile}\!\!\!x, \overset{2}{\smile}\!\!\!m)=0,$$

where the function $f(\overset{2}{x}, \overset{2}{m})$ *is a symmetric function* $f(\overset{2}{x}, \overset{2}{m})\equiv f(\overset{2}{m}, \overset{2}{x})$ *of its two arguments* x *and* m *of the second degree in each of them, characterized by the property that between the two roots* x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{2}{x_1}, \overset{2}{x_2})\equiv 0$$

holds as an identity in m.

I. Let $f(\overset{2}{x}, \overset{2}{m})\equiv a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$.

II. $\because f(\overset{2}{x_1}, \overset{2}{x_2})\equiv 0$, and x_1 and x_2 take the places of x and m,

$f(\overset{2}{x_1}, \overset{2}{x_2})\equiv a+h(x_1+x_2)+bx_1x_2+g(x_1{}^2+x_2{}^2)+f(x_1{}^2x_2+x_2{}^2x_1)+cx_1{}^2x_2{}^2\equiv 0$.

We are to investigate now the conditions on the parameters a, b, c, f, g, h that must hold in order that $f(x_1, x_2)$ may as a function of m be identically 0. The problem then is not necessarily to prove $f(x_1, x_2)\equiv 0$ for all equations, but to find all equations for which it is true that $f(x_1, x_2)\equiv 0$.

III. Let $Kx^2+Lx+M=0$ be the original equation; x_1 and x_2 the roots; then $-K(x_1+x_2)=L$; $K(x_1x_2)=M$.

Comparing this equation with I:

$$K\equiv g+fm+cm^2.$$

$$L\equiv h+bm+fm^2.$$

$$M\equiv a+hm+gm^2.$$

IV. Transform equation in I to this form:

$$a+h(x+m)+(b-2g)xm+g(x+m)^2+f(xm)(x+m)+c(xm)^2=0.$$

V. Also equation in II to this form:

$$a+h(x_1+x_2)+(b-2g)x_1x_2+g(x_1+x_2)^2+f(x_1x_2)(x_1+x_2)+c(x_1x_2)^2\equiv 0.$$

VI. Multiply V by K^2:

$$aK^2+hK^2(x_1+x_2)+(b-2g)K^2x_1x_2+gK^2(x_1+x_2)^2$$
$$+fK^2(x_1x_2)(x_1+x_2)+cK^2(x_1x_2)^2=0.$$

VII. VI becomes, by substituting for x_1x_2 and x_1+x_2 their values as given in III:

$$aK^2-hKL+(b-2g)KM+gL^2-fLM+cM^2\equiv 0$$

where K, L, M are given in terms of m in III.

Since VII is an identity in m, the coefficients of the different powers of m are each zero; $\therefore$ the condition in VII requires that five polynomials homogeneous in a, b, c, f, g, h of degree three shall be zero. Since there are six letters, there are five ratios; $\therefore$ there are five unknowns in five equations. This system of five cubic equations turns out to be extremely simple.

For, in VII, substituting for K, L, M their values involving m as given in III, collecting terms with reference to m, and using detached coefficients, we have:

1	m	m^2	m^3	m^4	
ag^2	$2afg$	af^2+2acg	$2acf$	ac^2	$\equiv aK^2$
$-gh^2$	$-(fh^2+bgh)$	$-(ch^2+bfh+fgh)$	$-(bch+f^2h)$	$-cfh$	$\equiv -hKL.$
$ag(b-2g)$	$(af+gh)(b-2g)$	$(ac+fh+g^2)(b-2g)$	$(ch+fg)(b-2g)$	$cg(b-2g)$	$\equiv(b-2g)KM$
gh^2	$2bgh$	$b^2g+2fgh$	$2bfg$	f^2g	$\equiv gL^2$
$-afh$	$-(abf+fh^2)$	$-(af^2+bfh+fgh)$	$-(f^2h+bfg)$	$-f^2g.$	$\equiv -fLM.$
a^2c	$2ach$	ch^2+2acg	$2cgh$	cg^2	$\equiv cM^2.$

Simplifying and letting c_0, c_1 ...... be coefficient of m^0, m^1 ......

$$c_0\equiv a\{(b-g)g+ac-fh\}=0.$$

$$c_1\equiv 2h\{(b-g)g+ac-fh\}=0.$$

$$c_2\equiv(b+2g)\{(b-g)g+ac-fh\}=0.$$

$$c_3\equiv 2f\{(b-g)g+ac-fh\}=0.$$

$$c_4\equiv c\{(b-g)g+ac-fh\}=0.$$

This means that given $f(x, m)=0$ as in I, then $f(x_1, x_2)\equiv 0$, if, and only if, either $a=2h=b+2g=2f=c=0$, or $(b-g)g+ac-fh=0$; the second alternative is *one* condition, homogeneous, of degree two, between the six homogeneous parameters. Therefore,

All quadratic equations of the form

$$a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$$

(in which the first member is a symmetric function $f(\overset{\overset{2}{\smile}}{x}, \overset{\overset{2}{\smile}}{m}) \equiv f(\overset{\overset{2}{\smile}}{m}, \overset{\overset{2}{\smile}}{x})$ *of its two arguments* x *and* m *of the second degree in each of them), whose parameters are related by the equation*

$$(b-g)g+ac-fh=0,$$

—and, apart from the relatively trivial equation

$$g(x^2-2mx+m^2)=0,$$

only those equations whose parameters are so related—are characterized by the property that between the two roots x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{\overset{2}{\smile}}{x_1}, \overset{\overset{2}{\smile}}{x_2}) \equiv 0$$

holds as an identity in m.

November 26, 1895.

QUADRATURE OF THE CIRCLE.

By WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

The problem of the quadrature of the circle, or what amounts to the same thing, drawing a straight line equal in length to the circumference of a given circle, occupied the attention of mathematicians at a very early date. Long before the time of Archimedes, geometers had attacked the problem with but one result: failure. And for more than twenty centuries mathematicians have been struggling with the problem. Many claimed to have solved it, but their analysis has been, in every case, found to be fatally defective. After centuries of attempt and failure mathematicians began to suspect that the problem might not admit of solution. James Gregory was the first to attempt a proof of the impossibility of the quadrature of the circle. In the opinion of Montucla he succeeded; but later mathematicians have not so decided. Not a score of years have passed since a rigid proof was given that the solution of the problem is really impossible under the conditions usually understood: that is, by the use of the rule and compass only.

It is well known from the geometry that the ratio of the circumference to the diameter of a circle is constant. This constant ratio is usually denoted by the Greek letter π, and it follows at once that if π is a number commensurable with unity that it can be constructed geometrically, and the problem is solved. Lambert, in a memoir presented to the Berlin academy in 1761, was the first to prove that π is incommensurable. Other proofs of this result have been given, especially by Hermite in Crelle's Journal, Vol. 76, which demonstration is reproduced in the Traite de Geometrie of Rouche and Comberousse, 4th edition. But this result, however interesting in itself, does not prove the impossibility of a geometrical construction of π. For example, the square root of 2 (or any non-quadrate number) is incommensurable but is easily constructed geometrically. The first real advance towards the solution of the problem was made by Hermite in 1873. Hermite succeeded in proving that the number e, the base of the Naperian system of logarithms is not only incommensurable but that it can not be a root of a rational algebraic equation of any degree whatever. Such a number is called transcendent. If the number π could be proved to be transcendent the vexed question of the quadrature of the circle would be settled once for all. For this problem requires to derive the number π by a finite number of elementary geometrical constructions. As two straight lines, or a straight line and a circle, or two circles, have not more than two intersections, these processes, or any finite combination of them, can be expressed algebraically in a comparatively simple form ; so that the solution of the problem of the quadrature of the circle would mean that π can be expressed as the root of an algebraic equation solvable by square roots. Hermite did not succeed in proving that π is a transcendent number, but in 1882 Lindemann extended Hermite's proof to include the number π as well as e among the transcendent numbers. Hermite and Lindemann's methods are complicated and obscure and many mathematicians attempted to simplify them. But not until very recently were these attempts rewarded with any degree of success. In January, 1893, Hilbert published a proof of the transcendency of e and π that reduces the problem to such simple terms as to be understood by mathematicians having only a moderate understanding of the principles of the calculus. Hilbert's proof depends upon certain properties of the definite integral

$$\int_0^\infty z^\rho\Big[(z-1)(z-2)(z-3)\ldots\ldots(z-n)\Big]^{\rho+1}e^{-z}dz,$$

suggested by the investigations of Hermite.

Immediately after the publication of Hilbert's proof, Hurwitz published a proof for the transcendency of e based on still more elementary principles. And finally, in May, 1893, Gordan published a proof of the transcendency of e and π in which only the known development of e^x in powers of x is made use of. This last proof is so simple that it should be introduced into university teaching everywhere. The numbers e and π are very intimately related, and before proceeding

to Gordan's proof of the transcendency of these two fundamental numbers I wish to give here a well known proof that e is incommensurable. We have

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|r}}+\frac{1}{\underline{|r+1}}\ldots\ldots$$

Assume, now, that e is a rational number $\frac{a}{r}$, where a and r are integers, and the fraction $\frac{a}{r}$ is in its lowest terms. Multiply this equation by $\underline{|r}$ and we see that all the terms preceding the term $\frac{1}{\underline{|r+1}}$ are integers. The series

$$\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}+\frac{1}{(r+1)(r+2)(r+3)}\ldots\ldots \text{ is less than}$$

$$\frac{1}{r+1}+\frac{1}{(r+1)^2}+\frac{1}{(r+1)^3}\ldots\ldots$$

That is less than $\frac{1}{r}$. Thus we have an integer equal to a proper fraction which is impossible. I will now give Gordan's proof of the transcendency of e and π. The proof for e will be seen to be an extension of the above well-known proof of the irrationality of e and apparently should have been discovered long ago.

The function e^x is defined by the series

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}\ldots\ldots$$

This, if we introduce the symbolic notation

$$\underline{|r}=h^r$$

and multiply by this quantity and any whole number c_r, passes into the form

(1) $$c_r h^r e^x=c_r(x+h)^r+c_r x^r u_r$$

in which $$u_r=\frac{x}{r+1}+\frac{x^2}{(r+1)(r+2)}\ldots\ldots$$

If $$\mu=\text{mod.}x$$

we have $$\text{mod.}u_r<e^{\mu};$$

and if we put $$u_r=q_r e^{\mu},$$

$$\text{mod.}q_r<1.$$

From (1) it follows:

$$c_r h^r e^x = c_r(x+h)^r + c_r x^r q_r e^{\mu}.$$

And
$$e^x \sum_{r=o}^{r=s} c_r h^r = \sum_{r=o}^{r=s} c_r (x+h)^r + e^{\mu} \sum_{r=o}^{r=s} c_r q_r x^r.$$

And if we put
$$\sum_{r=o}^{r=s} c_r\, x^r = \phi(x),\ \sum_{r=o}^{r=s} c_r\, q_r\, x^r = \psi(x);$$

$$(2) \qquad e^x\phi(h) = \phi(x+h) + e^{\mu}\psi(x).$$

If, now, there is an equation with integral coefficients, satisfied by the number e:

$$\sum_{k=o}^{k=n} c_k e^k = 0,$$

then from (2) we have

$$(3) \qquad 0 = \sum_{k=o}^{k=n} c_k \phi(k+h) + \sum_{k=o}^{k=n} c_k \psi(k) e^k.$$

If we choose for ϕ the function

$$\phi(x) = \frac{x^{p-1}}{\underline{|p-1}}[(x-1)(x-2\ldots\ldots(x-n)]^p$$

and for ρ a prime number greater than the numbers n and c_0, then will $\phi(k+h)$ in formula (3) become whole numbers.

$$\phi(h+1),\ \phi(h+2)\ldots\ldots\phi(h+n)$$

have the factor p, but

$$c_0\phi(h)$$

has not. If we let p increase, then ϕ and ψ become as small as we please, and formula (3) is impossible, and the number e transcendent.

If $i\pi$ is a root of an equation with integral coefficients:

$$(4) \qquad c(x-w_1)(x-w_2)\ldots\ldots(x-w_\rho) = 0,$$

then we have the formula

$$(5) \qquad (1+e^{w_1})(1+e^{w_2})\ldots\ldots(1+e^{w_\rho}) = 0.$$

If $c-1$ vanishing quantities are found among the sums

$$w_i \,;\; w_i+w_k \,;\; w_i+w_k+w_\lambda \ldots\ldots$$

and we designate those remaining by

$$a_1,\, a_2,\, a_3 \ldots\ldots a_n$$

and their moduli by

$$a_1,\, a_2,\, a_3 \ldots\ldots a_n,$$

formula (5) becomes

$$(6) \qquad 0=c+\sum_{k=1}^{k=n} e^{a_k}.$$

The symmetric functions of cw_k, as well as those of ca_k, are whole numbers. By formula (2) we have

$$(7) \qquad 0=c\phi(h)+\sum_{k=1}^{k=n}\phi(a_k+h)+\sum_{k=1}^{k=n} e^{a_k}\psi(a_k).$$

Let $$\phi(x)=\frac{(cx)^{p-1}}{\underline{|p-1}}c^{np}[(x-a_1)(x-a_2)\ldots\ldots(x-a_n)]^p,$$

and let p be a prime number greater than the numbers

$$c\,;\; n\,;\; c\,;\; c^n a_1 a_2 \ldots\ldots\; a_n.$$

The quantities $\phi(h)$ and $\sum_{k=1}^{k=n}\phi(a_k+h)$ are whole numbers :

$$\sum_{k=1}^{k=n}\phi(a_k+h)$$

contains the factor p, but $c\phi(h)$ does not.

If p increases the moduli of ϕ and ψ become as small as we please.

Formula (7) is impossible, and therefore π is a transcendent number.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

It is the object of this paper to put on record, once for all, general values for the centroid of areas, represented by the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the centroid of volumes represented by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1.$$

I. *Areas.* Let the density vary as $x^{k-1}y^{l-1}$, the thickness being constant.

$$\text{Then } \bar{x}=\frac{\iint x^k y^{l-1}dxdy}{\iint x^{k-1}y^{l-1}dxdy},\quad \bar{y}=\frac{\iint x^{k-1}y^l dxdy}{\iint x^{k-1}y^{l-1}dxdy}.$$

$$\therefore\ \bar{x}=\frac{\dfrac{a^{k+1}b^l}{\dfrac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k+1}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k+1}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}{\dfrac{a^k b^l}{\dfrac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}.$$

$$\therefore\ \bar{x}=\frac{\Gamma(km+m+\frac{k+1}{2})\,\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(km+\frac{k}{2})\Gamma(km+ln+m+\frac{k+l+1}{2}+1)}a\ldots\ldots\ldots\ldots(A).$$

$$\text{Similarly, } \bar{y}=\frac{\Gamma(ln+n+\frac{l+1}{2})\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(ln+\frac{l}{2})\Gamma(km+ln+n+\frac{k+l+1}{2}+1)}b\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

This gives the centroid of a quadrant of the area whatever be the values of k, l, m, n. Let $k=l=1$, so that the density is the same throughout the whole area.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+n+2)}{\Gamma(m+\frac{1}{2})\Gamma(2m+n+\frac{5}{2})}a,\quad \bar{y}=\frac{\Gamma(2n+1)\Gamma(m+n+2)}{\Gamma(n+\frac{1}{2})\Gamma(m+2n+\frac{5}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}a=\frac{4a}{3\pi},\quad \bar{y}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}b=\frac{4b}{3\pi}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}a=\frac{256a}{315\pi},\quad \bar{y}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}b=\frac{256b}{315\pi}.$$

Let $m=n=2$, then $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}a=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4a}{\pi},$$

$$\bar{y}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}b=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4b}{\pi}.$$

Let $m=0$, $n=1$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}a=\frac{16a}{15\pi},\quad \bar{y}=\frac{\Gamma(3)\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})}b=\frac{128b}{105\pi}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}a=\frac{a}{5},\quad \bar{y}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}b=\frac{b}{5},$$ the centroid of the area between the parabola and its tangents as axes.

Let the density vary as xy, so that $k=l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2m+1)\Gamma(3m+2n+\frac{7}{2})}a,\quad \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2n+1)\Gamma(2m+3n+\frac{7}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}a=\frac{8a}{15},\ \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}b=\frac{8b}{15}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}a=\frac{128a}{429},\ \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}b=\frac{128b}{429}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}a=\frac{2a}{9},\ \bar{y}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}b=\frac{2b}{9}.$$

Let the density vary as x the distance from the axis of ordinates so that $k=2$, $l=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+n+\frac{5}{2})}{\Gamma(2m+1)\Gamma(3m+n+3)}a,\ \bar{y}=\frac{\Gamma(2n+1)\Gamma(2m+n+\frac{5}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2m+2n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{2}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}a=\frac{3\pi a}{16},\ \bar{y}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}b=\frac{3b}{8}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}a=\frac{49.45\pi a}{2^{14}},\ \bar{y}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}b=\frac{63b}{384}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}a=\frac{5a}{14},\ \bar{y}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}b=\frac{3b}{28}.$$

Let the density vary as y the distance from the axis of abscissas so that $k=1$, $l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+2n+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(2m+2n+3)}a,\quad \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(m+2n+\frac{5}{2})}{\Gamma(2n+1)\Gamma(m+3n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}a=\frac{3a}{8},\quad \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}b=\frac{3\pi b}{16}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}a=\frac{63a}{384},\quad \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}b=\frac{49.45\pi b}{2^{14}}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}a=\frac{3a}{28},\quad \bar{y}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}b=\frac{5b}{14}.$$

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

NOTE on the Solution of Problem 53, by J. M. COLAW, A. M., Principal of High School, Monterey, Va.

As originally proposed the problem read "with 7% *annual interest* from date," while, it would seem by inadvertence, as reproduced in the November number, it reads "with *interest at* 7 *per cent.* from date."

I do not find the subject of "Partial Payments on Notes *with Annual Interest*" treated in any of our Arithmetics, except in Olney's *The Science of Arithmetic*, but there are doubtless other exceptions.

On page 191 of *Science of Arithmetic* it is stated that when partial payments are made on notes which bear *Annual Interest*, at other times than those at which the annual interest falls due, the method *usually adopted* is as follows:

Find the interest on the note for 1 year; and find also the amount of the payments made during the year from the times they were severally made to the end of the year.

If the payments amount to more than the interest due, take *their amount* from the amount of the note, and make the remainder a new principal.

But if the amount of the payments does not equal the interest due, the principal remains unchanged, and the amount of the payments is taken from the interest, the remainder being treated as deferred interest.

Proceed in this manner with each year till the time of settlement, the last period being that from the time the last annual interest fell due to the time of settlement.

Mr. Wilke's solution does not follow in all points the rule here laid down as *the usual one.* The question is, what is the rule in Ohio where the note was drawn?

55. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

How long will it take to count a million, in the following manner: the counter is to pronounce each syllable in the names of the successive numbers at the rate of one per second?

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

One, two, ..., nine—10 syllables—of the first order, are each pronounced 9 times in every hundred.

$\therefore$ The total for these is $9\times10\times10000=$ 900000.

The same, of the fourth order, are each pronounced 9000 times in every hundred thousand.

∴ The total for these is $9000\times10\times10=$	900000.
Ten, *eleven*, ..., *nineteen*—20 syllables—of the first and second orders, are each pronounced once in every hundred.	
∴ The total for these is $20\times10000=$	200000.
The same, of the fourth and fifth orders, are each pronounced 1000 times in every hundred thousand.	
∴ The total for these is $10\times20\times10000=$	200000.
Twenty, *thirty*, ..., *ninety*—17 syllables—of the second order, are each pronounced 10 times in every hundred.	
∴ The total for these is $10\times17\times10000=$	1700000.
The same, of the fifth order, are each pronounced 10000 times in every hundred thousand.	
∴ The total for these is $10\times17\times10000=$	1700000.
One hundred, *two hundred*, ..., *nine hundred*—28 syllables—of the third order, are each pronounced 100 times in every thousand.	
∴ The total for these is $28\times100\times1000=$	2800000.
The same, for the sixth order, are each pronounced 100000 times.	
∴ The total for these is $28\times100000=$	2800000.
Thousand is pronounced 999000 times.	
∴ The total for this word is	1998000.
The number of syllables in *one million* is	3.
The grand total is	13198003.

∴ 13198003 seconds=152 days, 18 hours, 6 minutes, 43 seconds, the time required.

[Chas. C. Crose, New Windsor, Maryland, sent in a solution of problem 49. The solution is by Algebra and is very good, but as the space in the MONTHLY is very limited even for unsolved problems, we reluctantly omit his solution. The published solution of problem 49 is not valuable because of its brevity, but because each step is the statement of a very elementary mathematical proposition, and hence can be comprehended by any one who has mastered these simple propositions. It is no discredit to a solution to be long if at the same time it is clear in its statements. EDITOR.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by Professor E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont.

Transform $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ into a product.

I. Solution by **ROBERT E. MORITZ**, B. Sc., Professor of Mathematics in Hastings College, Hastings, Nebraska; and **EDGAR KESNER**, Boulder, Colorado.

Adding and subtracting $4y^2z^2$, we have

$$(x^4+y^4+z^4-2y^2z^2-2x^2z^2+2y^2z^2)-4y^2z^2, =(x^2-y^2-z^2)^2-(2yz)^2.$$

$$\begin{aligned}(x^2-y^2-z^2)^2-(2yz)^2 &=(x^2-y^2-z^2-2yz)(x^2-y^2-z^2+2yz)\\ &=[x^2-(y+z)^2][x^2-(y-z)^2]\\ &=(x-y-z)(x+y+z)(x-y+z)(x+y-z),\\ \text{or } &-(y+z-x)(x+y+z)(x-y+z)(x+y-z).\end{aligned}$$

Similarly solved by *O. W. ANTHONY*, *J. SCHEFFER*, *C. D. SCHMITT*, *H. C. WILKES*, *B. F. YANNEY*, and *G. B. M. ZERR*.

II. Solution by **A. P. READ**, A. M., Clarence, Missouri.

By the method of the last solution, we get

$$[x^2-(y+z)^2][x^2-(y-z)^2]=(x-y-z)(x+y+z)(x-y+z)(x+y-z).$$

In a similar way by adding and subtracting first $4x^2z^2$ and then $4x^2y^2$, we obtain

$$(y-x-z)(y+x+z)(y-x+z)(y+x-z),$$

and

$$(z-x-y)(z+x+y)(z-x+y)(z+x-y).$$

Also solved in this way by *M. A. GRUBER*.

III. Solution by **ALFRED HUME**, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Lafayette County, Mississippi.

$x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ may be expressed as the determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}$$

which, as in Burnside and Panton's *Theory of Equations*, second edition, page 253, or, as in Weld's *Theory of Determinants*, pages 41 and 42, may be resolved into the factors

$$-(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

55. Proposed by **MARCUS BAKER**, M. A., U. S. Geological Survey, Washington, D. C.

Two right triangles ABC and ABD are so placed as to have one side $x(=AB)$ in common. From P the intersection of their hypotenuses is drawn c perpendicular to x. Knowing the hypotenuses $a=39$ feet and $b=25$ feet, and the perpendicular $c=12\frac{6}{7}$ feet, find x. Note this theorem

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2+x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c},$$

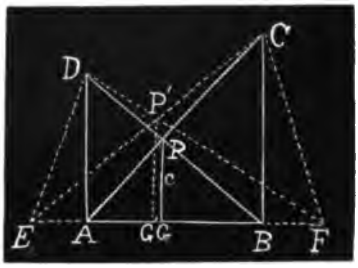
where m and n are the altitudes of the two triangles, respectively. Also find locus of P. Discuss the case when the triangles are general (not right angled).

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $AB=x$, $PG=c$, $AC=a$, $BD=b$, $CB=m$, $AD=n$.

From the triangles ABC and AGP, we get

$$m:c=x:x-GB\,\ldots\ldots\ldots\ldots(1).$$



From the triangles ABD and BGP, we get

$$n:c=x:GB\,\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating GB between (1) and (2), we get

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c}.$$

But $a=39$, $b=25$, $c=12\frac{6}{7}$.

$$\therefore \frac{1}{\sqrt{1521-x^2}}+\frac{1}{\sqrt{625-x^2}}=\frac{7}{90}. \quad \text{Let } 1521-x^2=y^2.$$

$$\therefore \frac{1}{y}+\frac{1}{\sqrt{y^2-896}}=\frac{7}{90}.$$

$$\therefore 7y^4-180y^3-6272y^2+161280y-1036800=0. \quad \therefore y=36,\ x=15.$$

For locus of P, let A be the origin. Using polar co-ordinates, we get

$$\tan\theta=\frac{\sqrt{a^2-x^2}}{x}, \text{ and } \frac{r\sin\theta}{x-r\cos\theta}=\frac{\sqrt{b^2-x^2}}{x}, \text{ for the equations to } AC \text{ and } BD.$$

The value of x from the first in the second gives

$$(a^2-b^2)(r\pm a)^2=(a^4\mp 2a^3r)\sin^2\theta. \quad \text{If } a=b,\ r=\pm\tfrac{1}{2}a.$$

For the general triangle, let $EF=x$, $P'G'=c$, $EC=a$, $DF=b$, $BC=m$, $AD=n$. Then from similar right triangles, we deduce the relation:

$$\frac{c}{n}(x-\sqrt{b^2-n^2})+\frac{c}{m}(x-\sqrt{a^2-m^2})=x.$$

II. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Let $BC=m$, $AD=n$, $PG=c$.

Then $\quad m:c::AB:AG. \quad \therefore AG=\dfrac{c.AB}{m};$

$$n:c::AB:BG. \quad \therefore BG=\frac{c.AB}{n}.$$

Adding, $AG+BG=AB=\dfrac{c.AB}{m}+\dfrac{c.AB}{n}$,

$$\text{or } 1=\frac{c}{m}+\frac{c}{n}, \text{ whence } \frac{1}{m}+\frac{1}{n}=\frac{1}{c}.$$

$$m=\sqrt{a^2-x^2}=\sqrt{1521-x^2}\ ;\ n=\sqrt{b^2-x^2}=\sqrt{625-x^2}.$$

Whence $\dfrac{1}{\sqrt{1521-x^2}}+\dfrac{1}{\sqrt{625-x^2}}=\dfrac{7}{90}$. Solving, $x=15$.

[SCHEFFER, SCHMITT.]

Putting $AG=x$, $PG=y$, we find from $\sqrt{b^2-\overline{AB}^2}:y=AB:x$, $\overline{AB}^2=\dfrac{b^2x^2}{x^2+y^2}$; and substituting this in $\dfrac{a^2-\overline{AB}^2}{y^2}=\dfrac{\overline{AB}^2}{(AB-x)^2}$, we get for the Cartesian equation of the locus $\dfrac{(a^2-b^2)x^2+a^2y^2}{y^2(x^2+y^2)}=\dfrac{b^2}{(b-\sqrt{x^2+y^2})^2}$.

Changing this into the polar equation by putting $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$, we obtain $r=\dfrac{b\sqrt{a^2-b^2\cos^2\theta}}{b\sin\theta+\sqrt{a^2-b^2\cos^2\theta}}$.

Also solved by *A. H. BELL* and *H. C. WILKES*. [See No. 24, Geometry, Vol. I, page 353, for another solution of a similar problem. Mr. Bell sends a trigonometrical solution, and says that his view of the problem in general is to have given a, b, c, and angles $ABC=BAD=\theta$, to find the base. EDITOR.]

ERRATA.—On page 359, line 16 of December issue, for "t_1+t_1" read t_1+t_2; page 360, line 7, in the denominator for "m_5" read m^5; page 360, under Case III., for "$-4m^5<n^2$" read $-4m^5>n^2$; in the last line on same page insert ½ before the second radical; and on page 361, line 3, of problem 52, for "(2)" read (3).

PROBLEMS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

63. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at same price per acre, which they sold for $286.90. A gained as much per cent. on his part as he had acres, B gained ⅝ as much per cent. on his part as A did, and C lost $9.10 on the cost of his part; the total net gain was 43$\frac{3}{10}$ per cent. How much land did each buy, and what did each receive per acre at the sale?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

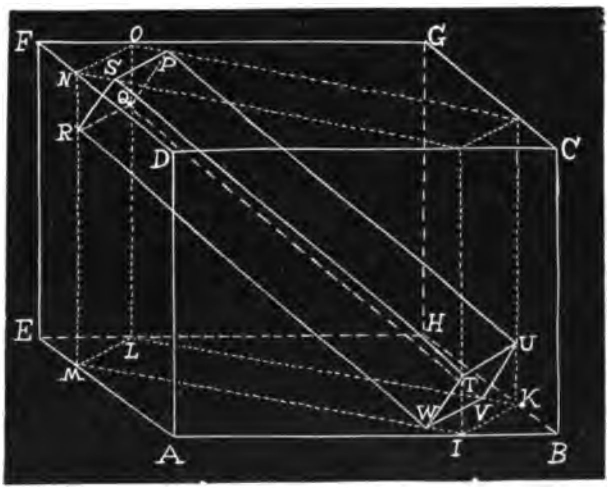
41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pennsylvania.

Find the length (x) of a rectangular parallelopiped $b=5$ feet, and $h=3$ feet, which can be *diagonally inscribed* in a rectangular parallelopiped $L=83$ feet, $B=64$ feet, and $H=50$ feet.

II. Solution by A. H. BELL, Hillsboro, Illinois, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $AB=L=83$ feet, $AE=B=64$ feet, $AD=H=50$ feet, $TUVW-P=$the required inscribed parallelopiped, $VW=b=5$ feet, $WT=h=3$ feet, $BK=z$, $IM=y$, and $WR=TS=UP=VQ=x$.

Then $KH=AM=B-z$, $BI=LE=(b^2-z^2)^{\frac{1}{2}}$, and $AI=HL=L-(b^2-z^2)^{\frac{1}{2}}$.



In the right triangle IAM, $IA^2+AM^2=IM^2$, or

$$[L-(b^2-z^2)^{\frac{1}{2}}]^2+(B-z)^2=y^2 \dots\dots\dots\dots (1).$$

In the similar triangles IAM and IBK, we have

$$AI:BK=IM:IK, \text{ or } L-(b^2-z^2)^{\frac{1}{2}}:z=y:b;$$

whence
$$yz=b[L-(b^2-z^2)^{\frac{1}{2}}] \dots\dots\dots\dots (2).$$

Solving (2) for y and substituting its value in (1) we have, after reducing and freeing of radicals,

$$4z^4-4Bz^3+[L^2+B^2-4b^2]z^2+2Bb^2z-(L^2-b^2)b^2=0 \dots\dots\dots (3).$$

Restoring numbers, we have

$$4z^4-2562z^3+10885z^2+3200z-171600=0\ldots\ldots\ldots\ldots\ldots(4).$$

Solving this equation by *Horner's Method*, we find $z=4$. $\therefore \sqrt{b^2-z^2}=3$.

$$\therefore \text{From (1) or (2), } y=100.$$

If in (3) we let $L=100$, $B=50$, and $b=3$ and solve the equation again for z, we find $z=2.750413+=TI$. $\therefore IW=1.5248$. $\therefore WR=109.4494693746751+$ feet, the required length of the parallelopiped.

Had we solved equation (2) for z and substituted its value in (1), we would have obtained an equation which would give the length of the rectangle $IMLK$, but it would require a great deal of work to free the equation of radicals. We shall now obtain such an equation, or formula.

Let $AB=L$, $BH=B$, $\theta=\angle AIM$, and $x=IM$. Then $AI=x\cos\theta$, $AM=x\sin\theta$, $IB=b\sin\theta$, $BK=b\cos\theta$.

$$\therefore \text{Area of } ABHE=2[\tfrac{1}{2}AI\times AM+\tfrac{1}{2}BI\times BK]+IM\times IK$$

$$=(x^2+b^2)\cos\theta\sin\theta=ab\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also

$$x\cos\theta+b\sin\theta=L\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$x\sin\theta+b\cos\theta=B\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Squaring (2) and (3) and adding the results, we have

$$x^2+c^2+4cx\sin\theta\cos\theta=L^2+B^2\ldots\ldots\ldots\ldots\ldots(4).$$

Equating $\sin\theta\cos\theta$ in (1) and (4), we have, after an easy reduction,

$$x^4-(L^2+B^2+2b^2)x^2+4LBbx-b^2(L^2+B^2-b^2)=0\ldots\ldots\ldots(5),$$

an equation which gives the length of the longest rectangle of given width which can be diagonally inscribed in a given rectangle.

[NOTE.—It is but justice to Mr. Bell to say that he was obliged to protest long and vigorously before he received a proper hearing to his claim that the published solution of Dr. Matz and Mr. Burleson is wrong. It was simply a case of that injustice commonly done to men when we believe them to be wrong and refuse to examine their claims. This problem was proposed a few years ago in the *School Visitor*, and at that time we solved the problem though we did not try to obtain the numerical result. When Dr. Matz and Mr. Burleson sent in their solution, it seemed to us on cursory examination to be obtained on the same plan pursued by us a few years ago. But after Mr. Bell had written to us on several different occasions, we offered to publish his solution that our readers might compare the results. But before doing so, we examined the published solution in the May No. Vol. II and found that it was wrong. The numerical calculation of $z=WR$ is due to Mr. Bell, as is also the last equation and the method of obtaining it. EDITOR.]

49. Proposed by J. C. WILLIAMS, Rome, New York.

Of all triangles inscribed in a given segment of a circle, with the chord as base, the isosceles is the maximum.

I. Solution by M. A. GRUBER, War Department, Washington, D. C., and A. P. REED, Superintendent of Schools, Clarence, Missouri.

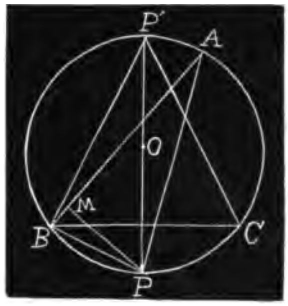
The bases being equal, the maximum triangle is the one having the greatest altitude.

In any segment of a circle, the greatest perpendicular that can be drawn to the chord, is the perpendicular to the middle of the chord. This perpendicular is the altitude of the isosceles triangle.

.·. The isosceles triangle is the maximum.

II. **Solution by J. M. COLAW, A. M., Superintendent of Schools, Monterey, Virginia, and E. KESNER, Boulder, Colorado.**

As the segment may be greater or less than a semi-circle, the general proof is for the circle. In the figure it is obvious that the isosceles triangle $P'BC$ is greater than any other triangle ABC, *as its altitude is greater.* Having the given chord as the common base, the *area* depends entirely on the altitude. But the isosceles triangle is a maximum both in *perimeter* and *area.*



Draw PM perpendicular to AB. Then the triangles APM, $P'PB$ are similar, and the diameter $P'P$ is $>AP$; .·. $P'B$ is $>AM$.

But $2AM=AB+CA$ (*Richardson and Ramsey's Modern Plane Geometry, pp.* 24, 131).

.·. $P'BC$ has the maximum perimeter.

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR.*

[NOTE.—This problem, with the addition that the isosceles triangle has the maximum perimeter, is Theorem 11, page 131, *Richardson and Ramsey's Modern Plane Geometry.* EDITOR.]

PROBLEMS.

54. **Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.**

Prove geometrically :

If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aD_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

55. **Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.**

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter th forms with the major axis ; it is required to find the length of this diameter.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26, AVERAGE AND PROBABILITY.

BY G. B. M. ZERR.

In reply to Dr. Martin, for whom I have the utmost respect, I have the following remarks to make. The problem that gives the result $\frac{1}{4}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$. In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states. The problem is as follows: Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another. In the problem under consideration the hypotenuse a is fixed and the right angle moves on the semi-circumference. In the first case the average length of one leg is $\int_0^a xdx \Big/ \int_0^a dx = \frac{1}{2}a$. In the second case the average length is $\int_0^a xds \Big/ \int_0^a ds = a/\pi$. In the first case the average area of all the triangles is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}dx \Big/ \int_0^a dx = \frac{1}{6}a^2$. In the second case the average area is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}ds \Big/ \int_0^a ds = \frac{a^2}{2\pi}$, where ds represents an element of arc. It is plainly evident that in the result $\frac{1}{6}a^2$ the leg does not and cannot change its direction or its average length would not be $\frac{1}{2}a$. In the second case it is constantly changing its direction and the right angle is moving on a semicircumference. The problem calls for a given hypotenuse and not one that is constantly changing its direction; hence the result $\frac{a^2}{2\pi}$ is the correct result.

DR. MARTIN'S RESULT IS NOT CORRECT.

F. P. MATZ.

Cause the problem to read: "Find the average area of all right-angled triangles having a given hypotenuse, *if an arm of the triangle vary uniformly;*" then Dr. Martin's result, $\frac{1}{6}h^2$, is perfectly correct.

Strip the problem of this italicised condition; that is, make the problem read as originally proposed; then the number of possible right-angled triangles is proportional to the *length* of the semicircumference of which the given hypotenuse is the diameter. This is *the correct* plan of solution. By *adhering* to *this* plan of solution, the correct result, $h^2/2\pi$, is obtained, regardless as to choice of independent variable.

Dr. Martin's result, $\frac{1}{6}h^2$, is *too great;* for he, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignores*

the consideration of the areas of practically an infinitude of right-angled triangles of which the major portion have one *rather small* acute angle—thus giving them areas *smaller* than $\frac{1}{4}h^2$.

Since not only all of Dr. Martin's *ignored* right-angled triangles, but *all possible* right-angled triangles, have been properly averaged in my solutions leading to (the result) $h^2/2\pi$, I repeat that this result is the correct one.

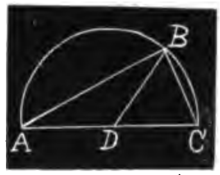
Mechanicsburg, Pa.

A REPLY TO DR. MARTIN'S NOTE.

BY THE EDITOR OF THIS DEPARTMENT.

I will say at first, that I too, have profound regard for Dr. Martin, and his opinion on a subject in which he was the pioneer writer in America should not be assailed simply for the sake of controversy.

His argument is entirely sound as to fact but not as to interpretation. It is true the triangles are *not* uniformly distributed on the semicircumference IF the number of triangles is to be obtained by varying one of the legs of the triangle. That this is true may be easily shown from the figure. Let $AC=a$, the hypotenuse, and $BC=x$, angle $BDC=\theta$. Then $x=a\sin\angle BDC=a\sin\frac{1}{2}\theta$. Differentiating, we have $\frac{dx}{d\theta}=\frac{1}{2}a\cos\frac{1}{2}\theta=\frac{1}{2}a\sqrt{\frac{1}{2}(\cos\theta+1)}$. $\therefore dx$ increases $\sqrt{\frac{1}{2}(\cos\theta+1)}$ times as fast as $d\theta$. When $\theta=0$, dx and $d\theta$ are increasing equally, and when $\theta=\frac{1}{2}\pi$, dx is increasing $\frac{1}{2}\sqrt{2}$ times as fast as $d\theta$. Hence it is evident that a greater number of triangles exist for a certain length of arc in the vicinity of the vertex of the semicircumference whose diameter is the hypotenuse, than for the same length of arc near the origin C when the number of triangles is made a function of one of the legs of the triangle, and therefore Dr. Martin's conclusion is sound if we grant his assumption, namely, that the number of triangles is a function of one of the legs of the triangle.



But this assumption is what we refuse to grant. We believe that there are other triangles that are to be interpolated in the series in order that the totality of the triangles may be obtained and that these interpolated triangles are found by making the totality a function of the semicircumference.

From this consideration, it is evident that Dr. Martin's result, $\frac{1}{4}a^2$, is greater than the result, $\frac{a^2}{2\pi}$, which we are defending. The reason is, that according to his interpretation, the triangles are most numerous when $x=\frac{1}{2}\sqrt{2}a$, that is to say, when the vertex of the triangle coincides with the vertex of the semicircumference. Hence the sum of the areas of the triangles ought to be greater than when only as many triangles are taken in one portion of the arc of the semicircumference as in any other.

If the radius DB is made to revolve with uniform velocity about the point

D and its extremity B be joined with the points A and C then the totality of triangles will be formed and they will be uniformly distributed on the semicircumference whose diameter is the hypotenuse.

The question is not whether the triangles are uniformly distributed or not but what method gives the *totality* of the series.

Drury College, January 27, 1896.

NOTES.

ERRATA. Professor Beman calls my attention to a manifest error in Professor Klein's paper which I translated for the December number. Vol. II, page 350, should give the series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots.$ The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots.$ $=\log_e 2$ instead of $\frac{\pi}{4}$.

D. E. SMITH.

Dr. E. A. Bowser writes: Should not problem 43 [Calculus] read $\int_0^1 \frac{x^{a-1}-x^{-a}}{1+x}\,\frac{dx}{\log x}$, as in Price's Calculus, Vol. II, page 120?

NOTE ON THE SOLUTIONS OF PROBLEM 45, PAGES 274-75.

BY ARTEMAS MARTIN, LL. D.

There is but *one* case in Problem 45, Geometry, *as proposed*. Only the circumscribing circle is required.

The final result may be expressed in the more simple form

$$R=\frac{abc}{2\sqrt{[abc(a+b+c)]}-(ab+ac+bc)}.$$

In the second solution, page 275, the equation

"$\cos BCA=\cos(BCA+BCO)$"

should be $\cos BCA=\cos(ACO+BCO)$.

EDITORIALS.

We shall be pleased to receive a catalogue from each of the schools and colleges where the MONTHLY is taken.

Charles De Medici, of New York City, writes, "Your magazine has certainly more merit than any other of the kind and ought to be well supported."

George W. Howe, Professor of Mathematics, State Normal School, Warrensburg, Mo., says, "The MONTHLY is a welcome visitor and I trust that you will continue it."

This number was delayed more than two weeks because of the failure of some proof reaching its destination. We feel confident that the March number will be mailed by the last of the month, and that thereafter the MONTHLY will appear regularly.

Cooper D. Schmitt, Professor of Mathematics, University of Tennessee, writes, "I enclose my subscription for the current year. I wish I had time to tell you how much I enjoy the MONTHLY and the good I get from it. It has caused me to study along certain lines that I had never before entered upon, and I feel that it does me an immense amount of good."

We are very thankful for the kind words that come from many of our contributors respecting the MONTHLY. That the work of getting out such a periodical every month is very arduous is not realized by all. That some errors creep into its pages is not surprising when every one thoroughly realizes the great work connected with the enterprise.

Professor E. P. Thompson, of Miama University, Oxford, Ohio, writes, "I send you with pleasure $2.00 for THE AMERICAN MATHEMATICAL MONTHLY for 1896. I get many a useful item, or point in discussion from it, and I hope you may prosper in the good work of putting into print the thoughts of the present workers in our beloved science."

Professor Thompson promises to contribute a paper on the "Mechanics of the Bicycle."

In *The Advance in Education* is an article, "A Class in Geometry under the Laboratory Plan," by Adelia R. Hornbrook, High School, Evansville, Ind. From this article we see that good practical use is made of the MONTHLY in the classroom. She says, "A group of boys, most of whom hope to go to the Polytechnic school, are working on a problem given them yesterday. They are much impressed with it, because it came out of the [*American*] *Mathematical Monthly.* They had never seen any mathematical publications except the text books,. and the *Monthly*, with its intricate diagrams, mysterious figures, and unfamiliar terms was a revelation to them." There are thousands of teachers in our High

Schools that could most profitably follow the writer of the above article's plan. Not necessarily that the MONTHLY be used but that the teachers of mathematics carry the spirit of the great living subject into their classrooms. There is no better way to do this than for every teacher to take some good magazine especially devoted to his favorite study.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus with Examples and Practical Applications. By J. W. Nicholson, A. M., LL. D., President and Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College. 8vo. Cloth, 256 pp. New York and New Orleans: University Publishing Co.

We have long been expecting this unique work on the Calculus as Col. Nicholson apprised us more than a year ago that he was preparing a work on the subject which he expected would create a stir among mathematicians. In this, I think he will realize his expectation, as his work is a great departure from the long beaten path of the traditional Calculus. None of the metaphysical speculations of Newton, Leibinz, Carnot, D'Alembert, Berkeley, Duhamel, Cavalieri, Marquis de L'Hopital, etc., are met with, in reading this book. The idea is simply an extention of mathematical principles without assuming vague metaphysical propositions.

The chief distinctions of this treatise is that, (1) it is based on the conception of *Proportional Variations*, (2) the treatment of dx as a variable, (3) a rigorous deduction of simple tests of absolute convergency, without recourse to the remainder in Taylor's formula, (4) an extension of the ordinary rules for finding maxima and minima, (5) a chapter on Independent Integration, (6) integration by independent coefficients, (7) the introduction of *turns* in curve tracing, and (8) a new proof of Taylor's formula.

The treatment of dx as a variable is the only rational way of viewing dx as a quantity at all. We do not think that Col. Nicholson has wandered too far from the usual method of treating the subject and we are sure the beginner in Calculus will hail the work with joy.

It is time for the Calculus to be treated on sound mathematical principles and not those of metaphysics. We very gladly recommend this new work to the favorable consideration of teachers and students desiring a good text book on the Calculus. B. F. F.

The Science Absolute of Space. Independent of the truth or falsity of Euclid's Axiom XI (which can never be proved *a priori*). By John Bolyai. Translated from the Latin by Dr. George Bruce Halsted, President of the Texas Academy of Science. Fourth edition. Vol. three of the Neomonic Series. Published at the Neomon, 2407 Guadalupe Street, Austin, Texas. Cloth, 71 pp. Price, $1.00.

Dr. Halsted has just got out the fourth edition of his translation of Bolyai's "Science Absolute of Space." The work is enriched by many interesting particulars concerning the lives of the celebrated author of the Non-Euclidean Geometry, Bolyai Janos, and his father, Bolyai Farkas. This little work is worth a careful reading at least once a year.

B. F. F.

Concrete Geometry for Beginners. By A. R. Hornbrook, A. M., Teacher of Mathematics in High School, Evansville, Indiana. 12mo. Cloth, 201 pp. Price, 75 cents. New York and Chicago: American Book Co.

This book is designed as an Introductory Course to the Study of Demonstrative Geometry. The author is a very thorough and efficient teacher of mathematics, and is intensely interested in the subject. The book is carefully and skillfully written, and can not be too highly recommended for the place it is designed to occupy. Were all students carefully instructed by the Laboratory Method, in Concrete Geometry, better results would be obtained while studying Demonstrative Geometry. B. F. F.

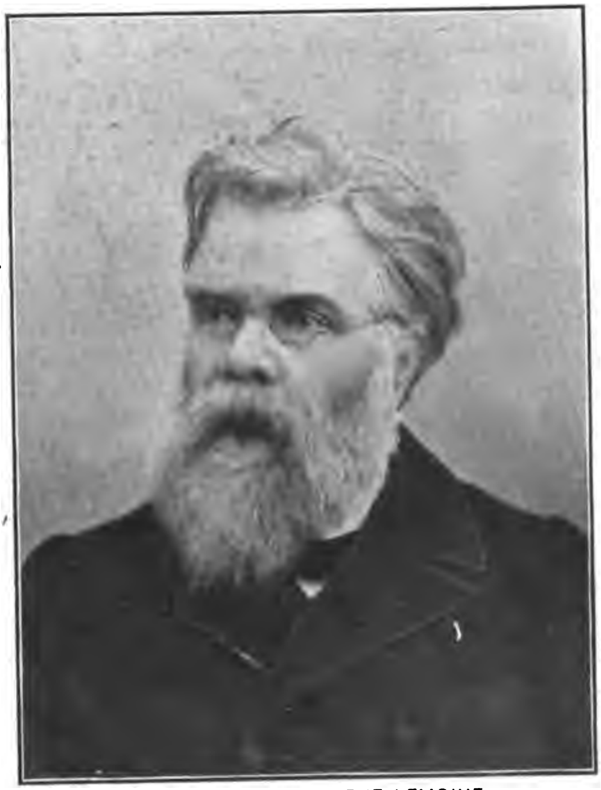
The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

During these months of extraordinary unrest in foreign politics, the *Review of Reviews* devotes its attention in large measure to international affairs. Its editorial department discusses matters in South Africa, the attitude of the great European powers, and the most recent phases of the movement among the nations for the arbitration of disbutes; the March number also contains a most timely article on "The Government of France and Its Recent Changes," by Baron Pierre de Coubertiu; "A Review of Canadian Affairs," by J. W. Russell, and a character sketch of "Cecil Rhodes, of Africa," by W. T. Stead. It can hardly be said that the *Review of Reviews* is narrowly provincial in its outlook on men and events!

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The price of this magnificent magazine has been so reduced as to make it possible for four-fifths of the homes of America to enjoy its choice reading. The March number is replete with the best literature of the present time.

See our club rate in December number and then give us your order for the *Review of Reviews* and the *Cosmopolitan.*



EMILE-MICHEL-HYACINTHE LEMOINE.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

Vol. III. MARCH, 1896. No. 3.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

It is proposed in these papers to give a more or less complete list of proofs, both new and old, of this celebrated and practical theorem. An attempt is made at classification based upon immediate principles used in the proofs. Due credit will be given in all known cases. A historical note will be appended to the completed list.

THEOREM.

The square described upon the hypotenuse of a right triangle is equivalent to the sum of the squares described upon the other two sides.

PROOFS.

I. Resulting from Linear Relations of Similar Right Triangles.

Let ABC be a $\triangle$ right-angled at C. Draw CD perpendicular to AB. There are thus three similar right triangles.

Letting $AC=b$, $BC=a$, $AB=c$, $CD=x$, $AD=y$, $BD=c-y$, we obtain the following proportions, with their resulting equations:

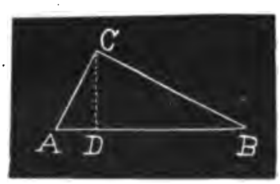


Fig. 1.

(1). $y : b :: b : c.$ $\therefore yc=b^2$ .... 1.

(2). $y : b :: x : a.$ $\therefore bx=ay$ .... 2.

(3). $b : c :: x : a.$ $\therefore cx=ab$ .... 3.

(4). $y : x :: x : c-y.$ $\therefore x^2=cy-y^2$ 4

(5). $y : x :: b : a.$ $\therefore bx=ay$ .... 2.

(6). $x : c-y :: b : a.$ $\therefore ax=b(c-y)$ .... 5.

(7). $c-y : a :: a : c.$ $\therefore c(c-y)=a^2$ .... 6.

(8). $c-y : a :: x : b.$ $\therefore ax=b(c-y)$ .... 5.

(9). $a : c :: x : b.$ $\therefore cx=ab$ .... 3.

From the nine different proportions, there are derived but six different equations, equation 2 being derived from proportion (2) or (5), 3 from (3) or (9), and 5 from (6) or (8).

It is evident that from no single equation can we determine the relation between a, b, and c, the sides of the given right $\triangle$.

It is also evident that there is but one set of twos which will give the relation desired, viz., equations 1 and 6. If we add these, member by member, we get directly $c^2=a^2+b^2$. Giving to this form the usual geometrical interpretation, we thus have one proof of the theorem. This, though in a different form, is one of the methods usually found in the books. It is credited to Legendre.

We now proceed to find combinations of threes, which will give the required relation. There are $\frac{6.5.4}{|\underline{3}}=20$ sets of three equations out of the six. But of these, four must be rejected, since they contain 1 and 6, which two alone prove the theorem, as already shown; also the following three sets, since in each set the equations are dependent: 1, 2, 3; 2, 4, 5; 3, 5, 6. There are, then, left the following thirteen sets, from each of which, if we eliminate x and y, we get $c^2=a^2+b^2$: 1, 2, 4; 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 3, 6; 2, 4, 6; 2, 5, 6; 3, 4, 5; 3, 4, 6; 4, 5, 6.

Of these thirteen sets, there are six that contain one equation each, derived from either of two proportions; six sets containing two each such equations; and one containing three. Therefore, including the proof already given, there are $1+6\times 2+6\times 2^2+2^3=45$ proofs, by this method.

II. Let ABC be a $\triangle$ right-angled at C. Draw a line perpendicular to AB from A, meeting BC produced, as at D.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=x$, $DC=y$, $BD=y+a$, and proceeding as in the preceding case, we find that this method also yields 45 different proofs. The details are left to be carried out by the reader.

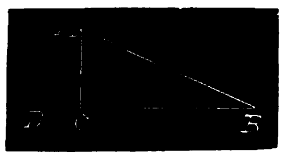


Fig. 2.

III. Let ABC be a $\triangle$ right-angled at C. Draw DE perpendicular to AB so that $DE=DC$. Then will $BE=BC$. $\triangle$ ADE is similar to $\triangle$ ABC.

Letting $AC=b$, $BC=a$, $AB=c$, $AE=c-a$, $DE=DC=x$, $AD=b-x$, we have:

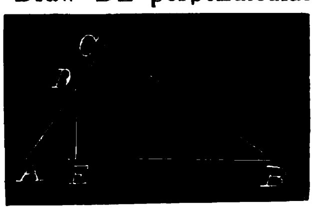


Fig. 3.

$$(1).\quad c-a : b :: x : a. \quad \therefore x=\frac{ac-a^2}{b} \quad\ldots\ldots\ldots\ldots 1.$$

$$(2).\quad c-a : b :: b-x : c. \quad \therefore x=\frac{b^2-c^2+ac}{b} \quad\ldots\ldots\ldots\ldots 2.$$

$$(3).\quad x : a :: b-x : c. \quad \therefore x=\frac{ab}{a+c} \quad\ldots\ldots\ldots\ldots 3.$$

From the three equations, it is evident that we may obtain three proofs by this method.

IV. Let ABC be a $\triangle$ right-angled at C. Extend AB to D making $BD=BC$. Draw a line perpendicular to AD at D, meeting AC produced as at E. Then will $CE=DE$; and $\triangle$ AED will be similar to $\triangle$ ABC.

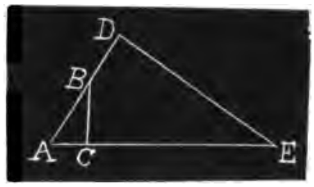


Fig. 4.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=c+a$, $DE=x$, $AE=x+b$, and proceeding as in the last case, we obtain three more proofs, making in all, thus far, 96 proofs.

In our next paper, we shall give a method whose results reach into the thousands.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from February Number.]

PROPOSITION XXIII. *If any two straights AX, BX (Fig. 27.) exist in the same plane, either they have (even in the hypothesis of acute angle) a common perpendicular, or prolonged toward either the same part, unless sometime at a finite distance one strikes upon the other, they mutually approach ever more toward each other.*

Proof. From any point A of AX is let fall to the straight BX the perpendicular AB. If BA makes with AX a right angle, we have the asserted case of the common perpendicular. But otherwise this straight makes toward one or the other part, as suppose toward the parts of the point X, an acute angle.

So in the aforesaid straight AX between the points A and X any points D, H, L are designated, from which are let fall to the straight BX the perpendiculars DK, HK, LK. If any one angle at the points D, H, L be acute toward the parts of the point A, it follows (from the preceding) that AX, BX will have a common perpendicular.

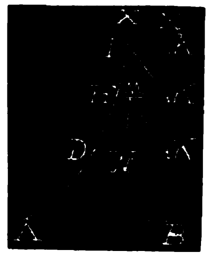


Fig. 27.

But if every angle of this sort be greater than acute; either some one will be right, and thus again we will have the asserted case of the common perpendicular, since all angles at the points K are supposed right; or all those angles toward the parts of the point A are obtuse, and therefore all therewith acute toward the parts of the point X, and so again I argue: Since in the quadrilateral $KDHK$ the angles at the points K are right, but the angle at the point D is acute, the side DK will be (from Cor. II. after P. III.) greater than the side HK.

In a similar way the side HK is shown to be greater than the side LK; and so always, comparing to each other perpendiculars from any ever higher points of AX let fall upon the other BX. Wherefore AX, BX mutually approach each other ever more toward the parts of the point X: Which is the second part of the disjunct proposition.

From all which follows that any two straights AX, BX, which exist in the same plane, either have (even in the hypothesis of acute angle) a common perpendicular, or produced toward either the same part, unless sometime at a finite distance one strikes upon the other, mutually approach each other ever more.

Quod erat etc.

Corollary I. Hence the angles toward the base AB will be always obtuse at each point of AX, from which is let fall a perpendicular to the straight BX: will be, I say, always obtuse, as often as those two AX and BX mutually approach each other ever more toward the parts of the points X; which indeed ought to be understood in a sane way, of course, of perpendiculars let fall before the aforesaid meeting, if perchance one should strike upon the other at a finite distance.

Scholion. I see indeed that it may be here inquired, in what way can be shown the existence of that common perpendicular, as often as any straight $PFHD$ (Fig. 28.) meeting two AX, BX in points F, and H, makes toward the same parts two internal angles AHF, BFH, not themselves indeed right, but nevertheless together equal to two rights. But behold that common perpendicular geometrically demonstrated.

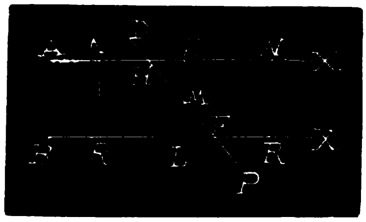


Fig. 28.

FH being bisected in M, perpendiculars MK, ML are let fall to AX and BX. The angle MFL will be equal (Eu. I. 13) to the angls MHK, which indeed is supposed to make up two right angles with the angle BFH. Moreover the angles at the points K and L are

right ; and again MF, MH are equal. Therefore (Eu. I. 26) so are the angles FML, HMK equal. Wherefore the angle HMK makes two right angles with the angle HML, since with this the angle FML (Eu. I. 13) makes two right angles. Therefore (Eu. I. 14) KML will be one continuous straight line, consequently the common perpendicular of the aforesaid straights AX, BX. Quod erat etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from February Number.]

The Construction of the Primitive Groups.

We have shown that all the intransitive and the non-primitive groups of a given degree, may be made to depend upon groups of a lower degree. We shall soon prove a similar property of the primitive groups.

It must however not be inferred that this will solve, in a satisfactory manner, the problem of constructing all the groups of a given degree. The elementary methods to which we have confined ourselves require a large number of trials if the degree is large. Some briefer methods will be given later but even these will only tend to make the construction of all the groups of a given degree practical for somewhat larger degrees.

It is not difficult to give general theorems which include all the groups of a given type, as, for instance, the theorem at the end of our discussion of the construction of the non-primitive groups ; but new types arise continually and no non-tentative method by means of which all the groups of any given degree may be found has yet been published.

We proceed to prove some theorems which apply to all transitive groups but are especially useful in the construction of primitive groups. Unless the contrary is stated the symbols G, g, and n will respresent respectively the group under consideration, its order and its degree.

Let us consider the transitive group G which contains the letters $a_1, a_2, \ldots\ldots, a_n$. The substitutions of G which do not contain a_1 (*i. e.*, those which replace a_1 by itself) may be represented by

$$s_1,\ s_2,\ \ldots\ldots,\ s_r \equiv G_1.$$

As every group must contain the identical substitution if the number of its

letters is finite and this is the only kind we are considering now, the minimum value of r is unity.

Since G is transitive it must contain some substitution s_{r+1} which replaces a_1 by a_2. We desire to find all the substitutions of G which have this property. If s_k is such a substitution then will

$$s_k s_{r+1}^{-1}$$

belong to the first line, since s_k replaces a_1 by a_2 and s_{r+1}^{-1} replaces a_2 by a_1 $s_k s_{r+1}^{-1}$ must leave a_1 unchanged. Hence we have the equations

$$s_k s_{r+1}^{-1} = s_\alpha \qquad (\alpha = 1, 2, \ldots\ldots r)$$

$$s_k = s_\alpha\, s_{r+1}.$$

Since the condition expressed by the last equation is sufficient as well as necessary it follows that there are just r different substitutions in G, which transform a_1 into a_2. Similarly there are exactly r substitutions in G which replace a_1 by a_3, etc. From this we see that the number of substitutions which replace a_1 by itself is equal to the number of those which replace a_1 by any other letter of G. We have imposed no condition upon a_1 which is not satisfied by each of the other letters so that the property which we have proved in regard to a_1 belongs to all the letters. That r has the same value for each of the letters follows from the following considerations :

If the substitutions of G which do not contain a_a are

$$s_1^1,\ s_2^1\ \ldots\ldots\ s_{r^1}^1 \equiv G_2,$$

then will

$$s_{r+1} G_2 s_{r+1}^{-1} = r^1 \text{ substitutions of } G \text{ which do not contain } a_1 \text{ and}$$

$$s_{r+1}^{-1} G_1 s_{r+1} = r \text{ substitutions of } G \text{ which do not contain } a_2.$$

From the first of these two equations we have $r' \overline{\gtrless} r$ and from the second $r \overline{\gtrless} r'$, hence $r' = r$. Similar remarks clearly apply to all the letters of G. We may embody the results at which we have arrived in the following

THEOREM: *The number of substitutions (r) of any transitive group (G), which do not contain any given letter, is equal to the number of substitutions which replace a letter by any required other letter of the group.*

Corollary I. $g = nr$, *i. e. the order of any transitive group is a multiple of its degree.*

*Corollary II. The average number of letters in all the substitutions of a transitive group of degree n is $n-1$.**

*Since every intransitive group may be resolved into transitive constituent groups whose separate elements enter an equal number of the substitutions of the intransitive group, the general statement of this corollary is as follows: *The average number of letters in all the substitutions of any group is $n-a$, n being the degree of the group and a the number of its transitive constituents.*

The last corollary may be proved as follows: Since G contains only $g \div n$ substitutions that do not involve a_α it must contain $g - g/n = \frac{n-1}{n}g$ that involve a_α. Hence all the g substitutions of G contain $n \times \frac{n-1}{n}g = (n-1)g$ letters.

From this corollary we may directly derive the following:

Corollary III. Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

Corollary IV. If the order of a transitive group exceeds its degree it must contain substitutions of a lower than the n^{th} degree and hence n conjugate subgroups $G_1, G_2, \ldots\ldots, G_n$ whose degree is at most $n-1$. These n subgroups need not all be distinct.

We may divide the primitive groups into two classes. (1) Those whose order is equal to their degree—the *regular* primitive groups—and (2) those whose order is b times their degree, where b is a positive integer larger than 1.

We proceed to consider the first one of these classes. Since the average number of letters in its substitutions is $n-1$ it must contain $n-1$ substitutions of the n^{th} degree, *i. e.* all its substitutions except unity are of the n^{th} degree.

If any one of these $n-1$ substitutions consists of two or more cycles all of these cycles will be of the same order, *i. e.* they will all contain the same number of letters, otherwise some power of this substitution would at the same time differ from identity and not contain all the letters of the group.

We proceed to prove the following

THEOREM: *Whenever a regular group contains a substitution (s) which contains more than one cycle it is non-primitive.*

Let $s = a_1 a_2 \ldots\ldots a_r, b_1 \ldots\ldots \ldots\ldots$ Some substitution of G (s_1) replaces a_1 by b_1. If we transform s with respect to s_1 we have

$$s_1^{-1} s\, s_1 = b_1 b_2 \ldots\ldots b_r \ldots\ldots$$

If we assume that

$$b_\alpha = a_\beta \qquad (\alpha, \beta \overline{\gtrless} r)$$

we have as a consequence that s_1 replaces a_α by a_β. This is also done by $s^{\beta-\alpha}$. Since only one substitution of G can perform this operation we have as a second consequence of the given assumption

$$s_1 = s^{\beta-\alpha}.$$

The latter of these transforms the cycle $a_1 a_2 \ldots\ldots a_r$ into itself and the former does not, the given assumption is therefore untenable and the cycle of b's must be distinct from the cycle of a's.

If these a's and b's do not include all the letters of G there must be some

substitution of G (s_2) which replaces a_1 by some new letter c_1. We now derive the substitution

$$s_2^{-1}s\ s_2 = c_1c_2\ldots\ldots c_2\ldots\ldots$$

We have already proved that these c's are all different from the a's. It remains to show that they do not include any b.

From $$c_\alpha = b_\beta$$

it would follow that s_2 replaced a_α by b_β and therefore that

$$s_2 = s^{\beta-\alpha}s_1.$$

This is impossible since the second member replaces the a's by the b's and the first replaces a_1 by c_1.

Continuing in this manner we must finally exhaust the letters of G and obtain the l distinct cycles

$$a_1a_2\ldots\ldots a_r,\ b_1b_2\ldots\ldots b_r,\ \ldots\ldots,\ l_1l_2\ldots\ldots l_r$$

where $lr = n$, the degree of G.

We proceed to prove that these cycles may be used as systems of non-primitivity. This is, of course, included in the proof that the substitutions composed of these cycles

$$a_1a_2\ldots\ldots a_r.\ b_1b_2\ldots\ldots b_r\ \ldots\ldots\ l_1l_2\ldots\ldots l_r \equiv t$$

is transformed into itself by all the substitutions of G.

Let s_α represent any substitution of G; we desire to prove that

$$s_\alpha^{-1}ts_\alpha = t.$$

If s_α replaces c_γ by b_β we have

$$s_\alpha = s_2^{-1}s^{\beta-\gamma}s_1.$$

The second member replaces $c_{\gamma+\rho}$ by $b_{\beta+\rho}$ where ρ satisfies the congruence

$$\gamma+\rho,\ \beta+\rho \equiv \delta \pmod r,\ (\delta = 1,\ 2,\ \ldots\ldots r).$$

Hence s_α must replace the c's in order by the b's in order. Since similar remarks apply to all the cycles it follows that s_α which is any substitution of G transforms t into itself and our theorem is proved.

By starting with the different cycles of G which contain the same letter we obtain different systems of non-primitivity for the same group.*

*Cf. Jordan, Traite des Substitutions, §75; and Netto, Theory of Substitutions (American Edition), §98.

From the last theorem we see that a regular group cannot be primitive unless it is generated by a single cycle involving a prime number of letters. Since such a group must be primitive we have the following

THEOREM : *The regular primitive groups and the prime numbers have a* 1,1 *correspondence ; i. e. for each prime number there is one regular primitive group and for each regular primitive group there is one prime number.*

[To be Continued.]

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Continued from February Number.]

II. VOLUMES. Let the density vary as $x^{h-1}y^{k-1}z^{l-1}$. Then

$$\bar{x}=\frac{\iiint x^h y^{k-1}z^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},\quad \bar{y}=\frac{\iiint x^{h-1}y^k z^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},$$

$$\bar{z}=\frac{\iiint x^{h-1}y^{k-1}z^l dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz}.$$

$$\therefore\ \bar{x}=\frac{\dfrac{a^{h+1}b^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\ \dfrac{\Gamma\left\{\frac{h+1}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n+1)\right\}\Gamma\left\{\frac{l}{2})2p+1)\right\}}{\Gamma\left\{\frac{h+1}{2}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}{\dfrac{a^hb^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\ \dfrac{\Gamma\left\{\frac{h}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n=1)\right\}\Gamma\left\{\frac{l}{2}(2p+1)\right\}}{\Gamma\left\{\frac{h}{3}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}$$

$$\therefore\ \bar{x}=\frac{\Gamma(hm+m+\frac{h+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(hm+\frac{h}{2})\Gamma(hm+kn+lp+m+\frac{h+k+l+1}{2}+1)}a \dots\dots\dots (C).$$

$$\bar{y}=\frac{\Gamma(kn+n+\frac{k+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(kn+\frac{k}{2})\Gamma(hm+kn+lp+n+\frac{h+k+l+1}{2}+1)}b \dots\dots\dots\dots (D).$$

$$\bar{z}=\frac{\Gamma(lp+p+\frac{l+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(lp+\frac{l}{2})\Gamma(hm+kn+lp+p+\frac{h+k+l+1}{2}+1)}c \dots\dots\dots\dots (E).$$

This gives the centroid of the eighth part of the volume whatever may be the values of h, k, l, m, n, p.

Let $m=n=p$, and also let the density vary as xyz so that $h=k=l=2$.

$$\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(6m+4)}{\Gamma(2m+1)\Gamma(7m+\frac{9}{2})}.$$

Let $m=0$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(1)\Gamma(\frac{9}{2})}=\frac{16}{35}$, for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$.

Let $m=1$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(\frac{9}{2})\Gamma(10)}{\Gamma(3)\Gamma(\frac{23}{2})}=\frac{2^{13}}{11.13.17.19}$,

for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$.

Let $m=\frac{3}{2}$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(6)\Gamma(13)}{\Gamma(4)\Gamma(15)}=\frac{10}{91}$,

for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$,

the centroid of the volume bounded by the positive portion of the co-ordinate planes.

Let $m=n=p$, and let the density be the same throughout the solid so that, $h=k=l=1$

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(2m+1)\Gamma(3m+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(4m+3)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)} = \frac{3}{8}$, for $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$.

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)} = \frac{21}{128}$, for $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$.

Let $m=\frac{3}{2}$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)} = \frac{3}{28}$, for $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1$.

Let $m=n=p$, and let the density vary as xy, so that $h=k=2$, $l=1$,

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(3m+\frac{3}{2})\Gamma(5m+\frac{7}{2})}{\Gamma(2m+1)\Gamma(6m+4)}, \quad \frac{\bar{z}}{c} = \frac{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}{\Gamma(m+\frac{1}{2})\Gamma(6m+4)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{\Gamma(1)\Gamma(4)} = \frac{5\pi}{32}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(1)\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(4)} = \frac{5}{16}, \text{ for} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(\frac{9}{2})\Gamma(\frac{17}{2})}{\Gamma(3)\Gamma(10)} = \frac{5.7.11.13.15\pi}{2^{20}}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(3)\Gamma(\frac{17}{2})}{\Gamma(\frac{3}{2})\Gamma(10)} = \frac{5.11.13}{2^{13}}, \text{ for} \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$$

Let $m=\frac{3}{2}$, — $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(6)\Gamma(11)}{\Gamma(4)\Gamma(13)} = \frac{5}{33}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(4)\Gamma(11)}{\Gamma(2)\Gamma(13)} = \frac{1}{22}, \text{ for} \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1.$$

Let $m=n=p$, and let the density vary as x so that $h=2$, $k=l=1$.

$$\therefore \frac{\bar{x}}{a}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(4m+3)}{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(4m+3)}{\Gamma(m+\frac{1}{2})\Gamma(5m+\frac{7}{2})}.$$

Let $m=0$, then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}=\frac{8}{15}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}=\frac{16}{15\pi}.$$

Let $m=1$, then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}=\frac{2^7}{3.11.13}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(7)}{\Gamma(\frac{3}{2})\Gamma(\frac{17}{2})}=\frac{2^{14}}{5.7.9.11.13\pi}.$$

Let $m=\frac{3}{2}$, then for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}=\frac{2}{9}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(9)}{\Gamma(2)\Gamma(11)}=\frac{1}{15}.$$

Thus we could multiply examples almost without number.
If $a=b$ we get another series of areas.
If $a=b=c$ we get another series of solids.
If $b=c$ or $a=c$ we get still another series of solids.
But formulæ (A), (B), (C), (D), (E) apply to them all.

One more example and we will proceed to the discussion of surfaces. Let the density vary as x^3y^2z, and let the equation to the surface be

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1,$$

so that $h=4$, $k=3$, $l=2$, $m=1$, $n=2$, $p=3$

$$\therefore \bar{x}=\frac{\Gamma(\frac{15}{2})\Gamma(\frac{43}{2})}{\Gamma(6)\Gamma(23)}a=\frac{5.7.9.11.13.23.29.31.37.39.41\pi a}{2^{50}},$$

$$\bar{y}=\frac{\Gamma(10)\Gamma(\frac{43}{2})}{\Gamma(\frac{15}{2})\Gamma(24)}b=\frac{5.9.29.31.37.41b}{11.2^{26}},$$

$$\bar{z}=\frac{\Gamma(\frac{21}{2})\Gamma(\frac{43}{2})}{\Gamma(7)\Gamma(25)}c=\frac{5.7.9.11.13.13.17.19.29.31.37.41\pi c}{2^{51}}.$$

The prodigious amount of work to accomplish this by the ordinary method would be impossible.

[To be Continued.]

THE ANGLE-SUM ACCORDING TO PLAYFAIR.

By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Mo.

In Playfair's Euclid, pages 295 and 296, there is given a short method for finding the angle-sum of a rectilineal triangle.

As the soundness of this method has been called in question by the Hyper-Space theorists, it is incumbent upon teachers of geometry to examine both the method itself and the criticisms to which it has been subjected.

John Playfair in treating of the angle-sum says—"It is of importance in explaining the Elements of Science, to connect truths by the shortest chain possible ; and till that is done, we can never consider them as being placed in their *natural order*.

The reasoning in the first of the following propositions is so simple, that it seems hardly susceptible of abbreviation, and it has the advantage of connecting immediately two truths so much alike, that one might conclude, even from the bare enunciations, that they are but different cases of the same general theorem, viz., That all the angles about a point, and all the exterior angles of any rectilineal figure, are constantly of the same magnitude, and equal to four right angles.

DEFINITION.

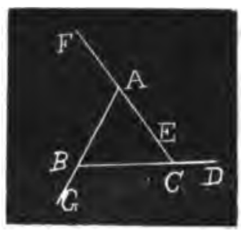
If, while one extremity of a straight line remains fixed at A, the line itself turns about that point from the position AB to the position AC, it is said to describe the angle BAC contained by the lines AB and AC.

Corollary. If a line turn about a point from the position AC till it come into the position AC again, it describes angles which are together equal to four right angles. This is evident from the second corollary to the fifteenth, 1.

PROPOSITION I.

All the exterior angles of any rectilineal figure are together equal to four right angles.

1. Let the rectilineal figure be the triangle ABC, of which the exterior angles are DCA, FAB, GBC; these angles are together equal to four right angles.



Let the line CD, placed in the direction BC produced, turn about the point C till it coincide with CE, a part of the side CA, and have described the exterior angle DCE or DCA.

Let it then be carried along the line CA, till it be in the position AF, that is, in the direction of CA produced, and the point A remaining fixed, let it turn about A till it describe the angle FAB, and coincide with a part of the line AB.

Let it next be carried along AB until it come into the position BG, and by turning about B, let it describe the angle GBC so as to coincide with a part of BC.

Lastly, let it be carried along BC till it coincide with CD its first position.

Then, because the line CD has turned about one of its extremities till it has come into the position CD again, it has by the corollary to the above definition described angles which are together equal to four right angles; but the angles which it has described are the three exterior angles of the triangle ABC, therefore the exterior angles of the triangle ABC are equal to four right angles.

2. If the rectilineal figure have any number of sides, the proposition is demonstrated just as in the case of a triangle. Therefore all the exterior angles of any rectilineal figure are together equal to four right angles.

Corollary 1. Hence, all the interior angles of any triangle are equal to two right angles. For all the angles of the triangle, both exterior and interior, are equal to six right angles, and the exterior being equal to four right angles, the interior are equal to two right angles."

In this demonstration of the angle sum Playfair evidently regards the method employed by him as legitimate, simple, direct and brief.

The Riemannian division of the Hyper-Space theorists assumes that a plane is the surface of an immense sphere, and that straight lines are curves that come back to their starting points, and, hence, raises the objection that such lines can not be slid along and then rotated as Playfair's demonstration requires.

This objection of the Riemannian School obviously rests on the false bottom that a plane is a spherical surface and that straight lines are curves. The foundation being insecure, that which is built thereon can not stand. The objection obliterates the distinction between spherical geometry and plane geometry.

If it be true that a plane is perfectly flat and that straight lines are devoid of curvature, the objection that we are considering is seen to have no force.

The Riemannian theorists tell us that for ought they know straight lines may be curves. They begin by doubting the truth of Euclid's second postulate—"That a terminated straight line may be produced to any length in a straight line"—and his Proposition XXXII, Book I. They are believers, also, as well as doubters. They believe that the angle-sum of a rectilineal triangle is greater than two right angles. They believe that if a straight line be extended it will ultimately return to the starting point.

The Euclidean geometers doubt these articles of Riemannian faith, and believe that the angle-sum of a rectilineal triangle is equal to two right angles, and that the longer a straight line is the further apart are its ends.

The Riemannians doubt what the Euclideans believe and believe what the Euclideans doubt. Those who undertake to teach both of these doctrines that contradict each other have failed to reckon with the logical laws of non-contradiction and excluded middle. We notice further that the Riemannian objection to Playfair's demonstration is in conflict with Proposition I of Lobatschewsky's

Theory of Parallels. Says the Russian Pangeometer—"A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place, if it goes through two unmoving points in the surface: (*i. e.*, if we turn the surface containing it about two points of the line, the line does not move)." These statements can not be made of any arc of any circle, and, hence, can not be made of Riemannian straight lines that are assumed to have constant positive curvature. What Lobatschewsky says respecting the straight line in his theorem I is inconsistent, of course, with his doctrine that the angle-sum is less than two right angles. But we are not quoting Lobatschewsky now to show that his theory is inconsistent with itself, but with that of Riemann.

Another objection to Playfair's demonstration is that a triangle drawn on a blackboard is not bounded by lines perfectly straight, since the surface of the board is uneven.

This objection does not hold against the triangle whose vertices are the three points A, B, and C in space and whose sides are destitute of curvature.

The geometer, whether he proceeds analytically or synthetically, naturally regards space as extending beyond himself on all sides without bounds, and between any two points A and B located therein he can draw an absolutely straight line with his mind, although he may be unable to do so with his hand. Some metaphysicians doubt these facts. What is it that they have not doubted? The function of a metaphysician, however, is to explain facts, not to doubt or discredit them.

Three points A, B, and C not in the same straight line may be located in trinally extended objective space and connected by the straight lines AB, AC, and BC. Hence, a rectilineal triangle in objective space is possible. When we say rectilineal triangle we do not mean a bogus triangle with wrinkled sides, but a genuine triangle with straight sides. When we say straight sides we do not mean wrinkled sides. The rectilineal triangle ABC of the geometer is perfect. His ability to cognize such a triangle is shown in the fact that he does cognize it. This fact, too, has been doubted. What a wonderful endowment that must be that enables man to people space with faultlessly perfect forms! This lofty power of intelligence in man, nay even his own doubt respecting it, differentiates him from the lower animals.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; and (2) prove that $\frac{n(n-1)(n+1}{24}$ is a whole number when n is odd.

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

(1). $1^2+2^2+3^2+4^2+\ldots\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}=$a whole number for all integral values of n.

(2). Let $n=2m+1=$an odd number for all integral values of m.

$\therefore \frac{(n-1)n(n+1)}{24}=\frac{m(m+1)(2m+1)}{6}=$same as (1).

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

As n and $n+1$ are consecutive numbers, one of them must be even and so divisible by two. But n must be of the form of $3p$, $3p+1$, or $3p+2$. If of the form of $3p$, it is divisible by three; if of the form $3p+2$, then $n+1$ or $3p+3$ is divisible by three; if of the form $3p+1$, then $(2n+1)$ becomes $6p+3$, and is divisible by three. Hence $n(n+1)(2n+1)$ is divisible by twice three, or six, whatever the value of n is.

2. $(n-1)n(n+1)$ of which the middle one is odd. One of every three consecutive numbers is always divisible by three: one of two consecutive *even* numbers is always divisible by four and the other by two. Hence $(n-1)n(n+1)$, when n is odd, is always divisible by $2\times3\times4$ or 24.

Also solved by O. W. ANTHONY, M. A. GRUBER, EDGAR KESNER, E. W. MORRELL, J. SCHEFFER, E. L. SHERWOOD, B. F. YANNEY, and G. B. M. ZERR.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College, Houston, Miss., and E. W. MORRELL, Department of Mathematics in Montpelier Seminary, Montpelier, Vermont.

$13^2.61^3=13^2.61^2.61=13^2.61^2(5^2+6^2)=13^2.61^2.5^2+13^2.61^2.6^2$.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $13^2.61^3=(p^2+q^2)^2(m^2+n^2)^3$, in which $p=3$, $q=2$, $m=6$, $n=5$. By decomposing into the sum of two squares, we find

$$(p^2+q^2)^2(m^2+n^2)^3=[m(p^2+q^2)(m^2-3n^2)]^2+[n(p^2+q^2)(3m^2-n^2)]^2=$$
$$[m(m^2+n^2)(p^2+q^2)]^2+[n(m^2+n^2)(p^2+q^2)]^2=$$
$$[m(p^2-q^2)(m^2-3n^2)\pm 2npq(3m^2-n^2)]^2+$$
$$[n(p^2-q^2)(3m^2-n^2)\mp 2mpq(m^2-3n^2)]^2=$$
$$[m(m^2+n^2)(p^2-q^2)\pm 2npq(m^2+n^2)]^2+[n(m^2+n^2)(p^2-q^2)\mp 2mpq(m^2+n^2)]^2,$$

making *six* sets of the sum of two squares.

Substituting the respective values of p, q, m, and n, we have

$$13^2.61^3=3042^2+5395^2=4758^2+3965^2=3810^2+4883^2$$
$$=6150^2+733^2=5490^2+2867^2=1830^2+5917^2.$$

Solved with these six sets of values by *A. H. BELL*, and with the five sets last in order by the *PROPOSER*. Also solved by *J. H. DRUMMOND*, *C. D. SCHMITT*, and *B. F. YANNEY*.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

I. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

We have $n^2+n=2y^2$. Putting $n=\frac{t-1}{2}$, we obtain $t^2-8y^2=1$. Since $t=3$, $y=1$, satisfy this equation, we have $t=\frac{1}{2}[(3+2\sqrt{2})^m+(3-2\sqrt{2})^m]$, where for m successive integral numbers must be chosen. The required values of n we then obtain from the relation $n=\frac{t-1}{2}$.

For $m=1$, 2, 3, 4, 5, 6, in succession, we find in order the corresponding values of $t=3$, 17, 99, 577, 3363, 19601, and $n=1$, 8, 49, 288, 1681, 9800.

II. Solution by A. H. BELL, Hillsboro, Illinois.

Let $\frac{n(n+1)}{2}=\square=y^2$ say; then clearing of fractions, multiplying by 4, and adding 1 to both members, etc., $(2n+1)^2=8y^2+1=\square=x^2$ say.

$\therefore n=\frac{x-1}{2}$. Again $x^2-8y^2=1$. The 1st convergent for the $\sqrt{8}=\frac{3}{1}=\frac{x}{y}$ or the solution of this celebrated equation and the value of x and y can be found, on page 53, Vol. I. of MONTHLY.

The general value for x is $x_{n+1}=2x_1\times x_n-x_{n-1}$, hence $x_0=1$, $x_1=3$, $x_2=6\times3-1=17$, $x_3=6\times17-3=99$, $x_4=6\times99-17=577$, $x_5=6\times577-99=3363$, and $x_6=19601$, etc.

$\therefore$ The required values of $n=1$, 8, 49, 288, 1681, 9800, etc.

III. Solution by the PROPOSER.

When $\frac{n(n+1)}{2}=\square$, one of the factors, n and $n-1$, is a square and the

other two times a square. Being known *one* of the values of n in $\frac{n(n+1)}{2}=\square$, the value next succeeding as well as the value just preceding can be found by the following formula which I deduced by inspection:

$$\frac{n(n+1)}{2}=\left(2n_1+1\pm3\sqrt{\frac{n_1(n_1+1)}{2}}\right)^2$$

in which n_1 is a known value of n. By inspection we find that when $n=1$, $\frac{n(n+1)}{2}=\square=1^2$. Now put $n_1=1$, and substituting in the formula, we obtain $\frac{n(n+1)}{2}=6^2$ or 0^2, 6^2 being the $\square$ next succeeding and 0^2 the square just preceding 1^2. From $\frac{n(n+1)}{2}=6^2$, we obtain $n=8(=2\times2^2)$, or $-9(=-3^2)$, and $n+1=9(=3^2)$ or $-8(=-2\times2^2)$. Now put $n_1=8$, and substituting in the formula, we get $\frac{n(n+1)}{2}=(35)^2$ or $(-1)^2$, the positive value being the next succeeding square and the negative value the one just preceding, the latter being the square with which we started. From $\frac{n(n+1)}{2}=35^2$, we find $n=49$ or -50, and $n+1=50$ or -49. By continuing this process, we find the first six positive integral values of n in $\frac{n(n+1)}{2}=\square$, to be 1, 8, 49, 288, 1681, and 9800.

IV. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

Let $n=p^2$ or p^2-1, since it must be a perfect power, or a perfect power less 1. Then $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}=a^2$; whence, $p^2\pm1=\frac{2a^2}{p^2}=2q^2$ ......(1).

Adding $2q^2+4pq+p^2$ to each member of equation (1), we have,
$2q^2+4pq+2p^2\pm1=4q^2+4pq+p^2$; or $(2q+p)^2\pm1=2(q+p)^2$..............(2).

Since equations (1) and (2) are the same in form, if we find one set of integral values for p and q in (1), we can then readily find succeeding values by (2). Now, for $p=1$, $q=1$. ∴ Other values are: 3 and 2; 7 and 5; 17 and 12; 41 and 29; 99 and 70; and so on. Then by formula $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}$, the first positive integral values of n are found to be 1, 8, 49, 288, 1691, 9800.

Also solved by *J. H. DRUMMOND, C. D. SCHMITT, H. C. WILKES, G. B. M. ZERR*, and the *PROPOSER*.

ERRATA. On page 368 of December issue, line 4, for "(10+2)" read (10^m+2); line 9, at end, for "B^2" read $B_1{}^2$; line 12, for "B^2" read $B_1{}^2$, and for

"$(B+1+A_1)$" read $(B+1-A_1)$; line 19, for "hypothenuse" read hypotenuse; line 22, leave out comma after 6; line 26, for "p, b, d," read p, d, b,; line 30, for "13, 14, 15," read 13, 15, 14; page 369, line 8, for "from" read for; line 25, for "the" read their; line 35, for "$a^{mn}+1$" read a^m+1; page 370, line 2, insert a comma before the sign of equality; and credit J. H. Drummond with a solution of No. 32.

NOTES, CRITICISMS, ETC., BY ARTEMUS MARTIN, LL. D.

On page 285 Mr. Adcock gives "An Equation for the sum of Squares equal a Square" which he says he has not seen published. I used the same method in the *Mathematical Magazine*, Vol. II., page 71, to find *three* square numbers whose sum is a square; and in a paper I had read at the last meeting of the American Association for the Advancement of Science I found in the same way *four* squares whose sum is a square. It is easily seen that the formula may be extended so as to find any number of squares whose sum is a square.

Note on Solutions of Problem 27, pp. 329-331.—In the *Mathematical Magazine*, Vol. II., No. 9, page 157, I have given a general method of finding any number (greater than two) of positive cube numbers whose sum is a cube, and on page 158 applied it to the case of five cubes, obtaining the set

$$6^3+11^3+13^3+18^3+20^3=26^3.$$

In Problem 42, p. 332, "$2a^2+2b^2-c^2+d^2$" should be $2a^2+2b^2=c^2+d^2$.

PROBLEMS.

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45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Penn.

Solve the equation $x^3+y^2=a^2$.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Give a general solution, finding such values of a and b in $x^2+x\sqrt{xy}=a$ and $y^2+y\sqrt{xy}=b$ as will make x and y integral.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each *sectoral triangle* of a regular inscribed dodecagon.

Solution by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let AOB and BOC be two adjacent sectors of the regular dodecagon. Let the dodecagon be determined by its apothem $=a$.

Let $\angle P_1OB=\theta_1$, $\angle P_2OB=\theta_2$, $OP_1=\rho_1$, $OP_2=\rho_2$.

Then area of triangle $P_1OP_2=$

$$\tfrac{1}{2}\rho_1\rho_2\sin(\theta_1+\theta_2).$$

And average area of triangle$=$

$$\Delta=\frac{1}{2}\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}\rho_1\rho_2\sin(\theta_1+\theta_2)d\rho_1d\rho_2d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}d\rho_1d\rho_2d\theta_1d\theta_2}.$$

$$OM=a\sec\left(\theta_1-\frac{\pi}{12}\right).$$

$$ON=a\sec\left(\theta_2-\frac{\pi}{12}\right).$$

Then $$\Delta=\frac{1}{8}a^2\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec^2\left(\theta_1-\frac{\pi}{12}\right)\sec^2\left(\theta_2-\frac{\pi}{12}\right)\sin(\theta_1+\theta_2)d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec\left(\theta_1-\frac{\pi}{12}\right)\sec\left(\theta_2-\frac{\pi}{12}\right)d\theta_1d\theta_2}.$$

The numerator may be written

$$\int_0^{\frac{1}{6}\pi}\left[\sec^2\left(\theta_2-\frac{\pi}{12}\right)\int_0^{\frac{1}{6}\pi}\sec^2\left(\theta_1-\frac{\pi}{12}\right)\sin\left[\left(\theta_1-\frac{\pi}{12}\right)+\left(\theta_2+\frac{\pi}{12}\right)\right]d\theta_1\right]d\theta_2.$$

The part under the last integral sign may be written, after expansion and some minor reductions,

$$\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})\tan(\theta_1-\frac{\pi}{12})d\theta_1+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\frac{d\theta_1}{\cos(\theta_1-\frac{\pi}{12})}$$

$$=\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\log_e\frac{1+\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})}{1-\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})},$$

$$=0+2\sin(\theta_2+\frac{\pi}{12})\log_e\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}.$$

$\therefore$ The numerator may be written:

$$2\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\sin(\theta_2+\frac{\pi}{12})d\theta_2.$$

The integral may be written

$$\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\left[\sin(\theta_2-\frac{\pi}{12})\cos(\theta_2-\frac{\pi}{12})\sin\frac{\pi}{6}\right]d\theta_2$$

=(after reductions similar to those above)

$$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\frac{d\theta_2}{\cos(\theta_2-\frac{\pi}{12})}=\frac{1}{2}\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)$$

$\therefore$ The numerator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

It may also be shown that the denominator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

$\therefore\ \Delta=\frac{1}{8}a^2.$

And the mean area of dodecagon=area of 12 such triangles $=\frac{3}{2}a^2$.

Solutions of this problem were received from G. B. M. Zerr and the Proposer, the latter furnishing two solutions.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by R. J. ADCOCK, Larchland, Warren County, Illinois.

When the sum of the distances of a point of a plane surface, from all other points, is a minimum, that point is the center of gravity of the plane surface.

IV. Discussion by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. Consider the following problem: Find a point within a plane surface such that the sum of the n^{th} power of the distances to all other points of the surface shall be a minimum.

$$S=\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^n dxdy.$$

For minimum—

$$\frac{dS}{dx_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(x_1-x)dxdy=0 \ \ldots\ldots\ldots\ldots(1),$$

$$\frac{dS}{dy_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(y_1-y)dxdy=0 \ \ldots\ldots\ldots\ldots(2).$$

(1) and (2) may be satisfied in several ways.

(*A*). The curve may be such that the integration in question performed over the surface reduce to zero.

(*B*). $\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=0$, or,

$$\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=C.$$

$$(C).\begin{cases}(x_1-x)dxdy=0, \text{ or} \int\int(x_1-x)dxdy=C_1\ \ldots\ldots\ldots\ldots\ldots\ldots(3),\\ (y_1-y)dxdy=0, \text{ and} \int\int(y_1-y)dxdy=C_2\ \ldots\ldots\ldots\ldots\ldots\ldots(4).\end{cases}$$

We shall only consider (*C*), as it is the only one which leads to the consideration of the center of gravity.

From (3) and (4), $x_1 = \dfrac{C_1 + \int\int x dx dy}{\int dx dy}$, and $y_1 = \dfrac{C_2 + \int\int x dx dy}{\int dx dy}$.

Therefore (x_1, y_1) is the center of gravity only when C_1 and C_2 are zero. For this condition to be fulfilled the first member of (3) and (4) must be evidently zero. From (1) and (2) we see that this will be true generally only when $n-1=0$, *i. e.*, $n=1$. Hence there can be no *general* proposition except for the sum of the squares of the distances.

II. $u = \int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^n dx dy.$

$$\frac{du}{dx_1} = 2n\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1}(x_1-x)dxdy=0,$$

$$\frac{du}{dy_1} = 2n\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1}(y_1-y)dxdy=0.$$

Whence $x_1 = \dfrac{\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1} x dx dy}{\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1} dx dy}$,

and $y_1 = \dfrac{\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1} y dx dy}{\int\int\left[(x_1-x)^2+(y_1-y)^2\right]^{n-1} dx dy}$.

For (x_1, y_1) to be identical with center of gravity $n-1$ must be zero.

III. Prof. Zerr in his proof has $S = \int P dA$. He then writes $\dfrac{dS}{dx_1} = \dfrac{(x-x_1)dA}{D}$. He should have written $\dfrac{dS}{dx_1} = \int\dfrac{x-x_1}{D}.\,dA$; the integration was with respect to A and the differentiation with respect to x_1, and the two do not destroy each other.

IV. The proposition fails to hold for the simplest case imaginable, an indefinitely narrow rectangle, or straight line. Thus let AB be a straight line, P any point on that line. $AP=a$, $AB=l$, $PQ=x$. Then the sum of distances from

A ——— P ——— Q ——— B

$P=S=\int_{-a}^{l-a} xdx=\frac{1}{2}[l^2-2al]$. $\frac{dS}{da}=-2l=0$ for minimum, *i. e.*, $l=0$ which is an absurdity. The sum of squares a minimum *will* hold in this case.

V. The same proof that Prof. Zerr gives will hold for *any* power of the distance, which proposition is highly improbable.

31. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

In order that a vertical cylindric stalk may be severed by a blow of minimum force, the stalk must be struck at what inclination by a sharp wedge-shaped blade?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let $f\phi(\theta)$=the force necessary to sever a unit of area, where θ is the inclination to the horizon. Let r=radius of stalk.

$\therefore$ The area of section made in cutting is $\pi r^2\sec\theta$, the area of an ellipse with semi-axes r and $r\sec\theta$. $\therefore$ $\pi r^2 f\sec\theta\phi(\theta)$=a minimum. This can be made a minimum when $\phi(\theta)$ is known. If $f\phi(\theta)=a+b\cos^2\theta$, then $\theta=\sin^{-1}\sqrt{1-\frac{a}{b}}$.

32. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

Intermittent reflections of flashes of light on a clear sky after dark, indicated a storm was progressing *below* the horizon. Refraction of 34′ on the horizon, brought the upper edge of the storm-cloud up to the horizon, and was just visible. How far off was the storm if the cloud was one mile above the earth?

I. Solution by the PROPOSER.

In the plane triangle ABC, let C be the center of the Earth, A the place of the observer, and B that of the cloud. Then AC=Earth's mean radius=3959 miles, $=b$, BC=3960 miles, $=a$, $AB=c$, the required distance. The angle BAC=the nadir distance of the cloud, being $90^\circ-34'=89^\circ 26'=A$. Then

$\sin B=\frac{b\sin A}{a}$. $\therefore$ $B=88^\circ 35'36''$, and $180^\circ-(A+B=1^\circ 58'24''=C$, and

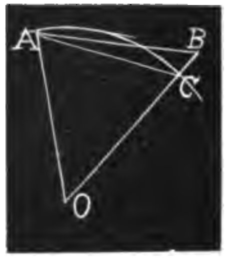
$$c=\frac{b\sin C}{\sin B}=\frac{a\sin C}{\sin A}=136.367 \text{ miles.}$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let A be the position of the observer, B the cloud, O the center of the earth, R=mean radius of the earth=3958 miles.

$$\therefore\ AC=2R\sin\tfrac{1}{2}AOC.\quad \angle ACB=\frac{\pi}{2}+\tfrac{1}{2}AOC,\quad \angle BAC=\tfrac{1}{2}AOC-34'.$$

$$\therefore AC=\frac{BC\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$



$$AB=\frac{BC\cos\frac{1}{2}AOC}{\sin(\frac{1}{2}AOC-34')}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$BC=1$ from (1). $2R\sin\frac{1}{2}AOC=\dfrac{\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')}$.

$\therefore \cos(AOC-34')=\dfrac{(1+R)\cos 34'-1}{R+1}$. $\therefore$ $AOC=1°58'16''$. From (2) $AB=136.778$ miles.

ERRATA. On page 56, second line from top, for "$-2562Z^3$" read $-256Z^3$; sixth line from top for "$Z=2.750413$" read $Z=2.750458368$, and for "$WR=1.5248$" read $WR=1.2963390864$; and in Note, second line from bottom, for "$z=WR$" read $x=WR$.

PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42°30'$ north$=\lambda$, at what angle with the horizon, will the sun rise, its declination$=22°$ north$=\delta$?

39. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minutes, makes 6000 vibrations in 1 hour and 9½ minutes. What is the length of the pendulum? And what length should it have to keep true time?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The Origin of π.—At least from the time of Archimedes π has stood for the number expressing how many times the diameter the circumference is. It is the initial letter of the Greek word περιφέρεια, meaning periphery. If the diameter is taken as a unit, then π stands for the periphery, or circumference. This is in reply to query of Lottie Smith in December Number.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

An Expression for π.—Though the result is not new, I have not seen it developed as follows:

$$\text{Since } e^{i\pi}=-1, \quad \therefore\ i\pi\log e=\log(-1).$$

$$\therefore\ \pi=\frac{\log(-1)}{\sqrt{-1}}.$$

BENJ. F. YANNEY.

Referring to the Note of R. Greenwood in the December Number, I would state that (1) probably the other root was infinite. Thus the equations $x^2-y^2=5$ and $x+y=5$ have roots $x=3$ or ∞, and $y=2$ or $-\infty$. (2) The proof that imaginary roots enter in pairs assumes that all the coefficients are real. The equation $x^2-bix=a^2-abi$ has roots: a and $-a+bi$ but its coefficients are not all real. (3) The equation $\sqrt{2x_2-2}-(3x-5)=0$ or A must be multiplied by $\sqrt{2x^2-2}+(4x-5)=B$ or 0 to give a quadratic equation. The given equation is not of the second degree as Mr. Greenwood seems to imply but of the $\frac{1}{2}$ degree. An infinite number of equations can be written that have no roots at all: for instance, $2x-5+\sqrt{x^2-7}=0$ (or ?). This when combined with its congeneric $2x-5-\sqrt{x^2-7}=?$ or 0 gives a quadratic; the last expression takes both roots leaving no root for the first form. The demonstration that every equation has a root referred to equations free from surds. H. C. WHITAKER,
Manual Training School, Philadelphia.

Another Reply: By squaring the equation, we get $2x^2-2=(3x-5)^2\ldots(1)$ $=2x^2-2-(3x-5)^2=0$, which is equivalent to transposing the member $3x-5$, and then multiplying the equation by $\sqrt{2x^2-2}+(3x-5)=0$. By doing this we have really introduced a new equation, which is satisfied for $x=\frac{9}{7}$.

Observe that (1) is satisfied for both $x=3$, and $x=\frac{9}{7}$, for it contains both the original equation and the one introduced by the questionable operation of squaring. Therefore, if the given equation means the positive root of $(2x^2-2)=3x-5$, then 3 is the only value of x that will satisfy it.

If $\pm\sqrt{2x^2-2}=3x-5$, both 3 and $\frac{9}{7}$ will.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

Note on Solution IV., Page 190. It does not follow that triangies AEL and ADK are equal because the triangles AEL and ADK are similar respectively to AFN and AGM, and the solution fails.

I would like to see a direct proof of this problem. It is said that the mathematician Todhunter failed to produce a direct proof of it.

GEORGE LILLEY,
394 Hall Street, Portland, Oregon.

Problem by Euler. One answer is given, but he adds there are many more. Legendre asks for a general solution as Euler's solution is lost: and he says such

a solution would be very much prized by mathematicians, if it could be given.

1st. The sum of the squares of each, horizontal, vertical, or diagonal rows shall be equal,—10 conditions.

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

68	-29	41	-37
-17	31	79	-32
59	28	23	61
-11	-77	8	49

=Euler's Numbers.

2nd. The sum of their products taken two and two=0, taking any two rows, horizontal, vertical, and diagonals,—12 conditions.

$$AE+BF+CG+DH=0=AD+FG+KL+NQ, \text{ etc.}$$

HILLSBORO, ILL., MATHEMATICAL CLUB.

Note on No. 4—Miscellaneous. In regard to No. 4 Miscellaneous, I had worked the problem with the assumption made by Prof. Hume, but rejected my solution, as on further thought I did not consider the assumption warranted. The constantly changing curvature carries with it a change in actual contact as well as in the amount ground off, which I have not been able to analyze. The assumption made would seem to apply if the stones were kept pressed together with such a force as would not yield, and would cause the particles to overlap for a constant distance. This also would require a constantly changing pressure or adjustment.

I should like to ask whether any one knows of a principle which will apply to the effect of friction in a case of this kind.

C. W. M. BLACK,
Wesleyan Academy, Wilbraham, Mass.

Query. Is a man who writes for publication in a Mathematical Magazine a "Note on Helmholtz's use of the terms 'Surface' and 'Space' as identical in meaning", properly to be considered sane?

Again when he asks "Does the 'immortal' Helmholtz in his Lectures on the—'Origin and Significance of Geometrical Axioms'—use the terms 'surface' and 'space' as identical in meaning?" since Helmholtz never delivered any lectures under this title, would it be sane to attempt to answer?

G. B. HALSTED.

The equation from Bell's Algebra, quoted by Mr. Greenwood, (MONTHLY, Vol., p. 372) is consistent if the radical be given the double sign. The equation should be

$$\pm\sqrt{2x^2-2}=3x-5.$$

The value $x=3$ belongs to the upper sign, $x=\frac{9}{7}$ to the lower.

WM. E. HEAL.

The answer to query (Monthly, Vol. II., p. 247) is not satisfactory. It is true "We *have* no method of finding the cube root by means of a compass" [and rule] but that does not prove the *impossibility* of a solution. What I wish, is a rigorous proof of the impossibility of expressing the roots of a cubic equation by a geometrical construction.

WM. E. HEAL.

Concerning the value of *factorial zero*, Chrystal says (Text Book of Algebra, Part II., page 4) "Strictly speaking 0! has no meaning. It is convenient, however, to use it, with the understanding that its value is 1; by so doing we avoid the exceptional treatment of initial terms in many series."

WM. E. HEAL.

IS THERE MORE THAN ONE ILLIMITABLE SPACE?

The Metageometers assume without proof that there are many varieties of space, differing in curvature, in the number of dimensions and in extent. Is their assumption axiomatic or does it need proof? Is it not really inconsistent with the hypothesis that space is everywhere and illimitable?

The Metageometers concede that the space that contains our Universe may for aught they know to the contrary, be trinally extended, i. e., through any point of it, whatever, three straight lines may be drawn mutually at right angles to each other. Notwithstanding this concession, they assume that there are two varieties of space at least, the number of whose dimensions is less than three.

They call a surface a variety of space that has *two* dimensions, and a line a variety of space that has *one* dimension.

The Euclidian geometers locate all their lines and surfaces in the one, trinally extended, illimitable space. They do not regard these lines and surfaces as distinct varieties of space that may be classed under an n-fold species.

Some of the Metageometers call a line one dimensional space, and a surface two dimensional space, apparently with the expectation that this ambiguous use of the word space will somehow assist them in ascending from our tridimensional space to a hypothetical one of four dimensions, and from that to one of five dimensions, and so on. This is certainly a most hazardous enterprise that they have undertaken. They are attempting to scale the transcendental heights of Hyper-space with an analogical ladder constructed out of defective timber. The two bottom rounds—one dimensional space and two dimensional space—are unable to endure the strain put upon them. We do not mount to trinally extended space from surfaces, nor to surfaces from lines. But we start with trinally extended space and in it locate surfaces and lines.

Successful ascent cannot be made from tridimensional space to fourdimensional space.

1st.—Because no one knows or can know the direction from 3-fold space to 4-fold, even if the latter exists.

2nd.—Because no one knows or can know that 4-fold space exists for the reason that the fundamental laws of thought are violated in every effort of the mind to cognize it. Legitimate thinking cannot proceed in violation of logical law, but stultification may do so. The so-called "generalized space" of the Metageometers is believed to be the joint product of pseudo-generalization, pseudo-analogical reasoning, and pseudo-analytical interpretation.

JOHN N. LYLE.

BOOKS AND PERIODICALS.

Trigonometry for Schools and Colleges. By Frederick Anderegg, A. M., Professor of Mathematics, and Edward Drake Roe, Jr., A. M., Associate Professor of Mathematics in Oberlin College. 8vo. Cloth, 108 pp. Boston : Ginn & Co.

This little work is a decided improvement over most modern treatises on trigonometry. It treats the subject with clearness and accuracy and leads the student to an easy acquaintance with modern higher mathematics. A number of new features are introduced. This is the first book we have yet seen in which it is shown that Plane Trigonometry is a special case of Spherical Trigonometry. Many other subjects of equal interest and importance are discussed. The authors deserve much credit for this original and unique work.

B. F. F.

An Elementary Treatise on Rigid Dynamics. By W. J. Loudon, B. A., Demonstrator in Physics in the University of Toronto. 8vo. Cloth, 236 pp. Price, $2.25. New York : Macmillan & Co.

This is a most excellent treatise on Rigid Dynamics. The subjects treated are made very clear and the student is still further aided in grasping those complex and difficult principles by very beautiful and accurate diagrams. Any student who has mastered the calculus can take up this work without any difficulty. At the close of each subject is a list of problems. The book closes with 306 problems all of which are very interesting to the student of dynamics. Some of these excellent problems will appear in future numbers of the MONTHLY.

B. F. F.

Notations de Logique Mathématique. Par G. Peano, Professeur d'Analyse infinitésimale à l'Université de Turin. Introduction au Formulaire de Mathématique Publie par la *Revista di Matematico*, Turin. Pamphlet, 52 pages.

A very interesting and valuable treatment of the notations of mathematical logic.

B. F. F.

Periodico di Mathematica. By L'Insegnamento Secondario. Pubblicato per cura di Aurelio Lugli, Professor di matematica nel R. Istituto tecnico di Roma.

The January-February number of this magazine contains a number of important papers and the solutions of 7 problems. B. F. F.

El Progreso Mathemático Periodico de Mathemáticas Puras y Aplicadas. Director D. Zoel G. de Galdeano, Catedrático de Geometria Analica en la Universidad de Zaragoza.

In this journal are published problems which are proposed by the best mathematicians in the world. The solutions are illustrated by beautiful diagrams. B. F. F.

Annals of Mathematics. Ormond Stone, Editor, Office of Publication, University of Virginia. Bi-monthly. Price, $2.00.

The September (1895) number contains the following articles: On the Improbability of Finding Shoals in the Open Sea by Sailing over the Geographical Positions in which they are Charted. By Mr. G. W. Littlehale. Note on the Congruence $2^{4n}\equiv(-)^n\,(2n)!/(n!)^2$, where $2n+1$ is a prime. By Prof. Frank Morley. Equations and Variables Associated with the Linear Differential Equation. By Dr. Geo. F. Metzler. The Calculus of Variations. By Dr. Harris Hancock. B. F. F.

March Monthly Magazine Number of The Outlook. Price, $1. per year in advance. The Outlook Company, 13 Astor Place, New York.

The illustrated monthly "Magazine Number of *The Outlook* for March has nearly fifty pages of reading matter, and more illustrations than any of the previous issues. Dr. R. L. Dickinson writes as an expert on hygienic and practical aspects of "Bicycling for Women," with cuts showing just what is right and wrong about women's riding; Edward Everett Hale tells of the "Higher Life of Boston;" there is a pleasant "Spectator" talk about picturesque New Orleans; Charleston of to-day is compared with its ante-bellum life in Mr. W. J. Abbot's "From Atlanta to the Sea;" Martin Luther is the subject of a fine article by professor Harnack, the great German theologian; and Mr. A. R. Kimball has a readable article about Penzance and the Newlyn school of artists. All these articles are fully illustrated. Ian Maclaren's novel gains in interest and humor.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The General of the Army, the General commanding the U. S. Corps of Engineers, Vice-Pres. Webb of the New York Central, and John Jacob Astor, compose *The Cosmopolitan Magazine's* Board of Judges to decide the merits of the Horseless Carriages which will be entered in the May trials, for which the *The Cosmopolitan* offers $3000 in prizes. This committee is undoubtedly the most distinguished that has ever consented to act upon the occasion of the trial of a new and useful invention. The interest which these gentlemen have shown in accepting places upon the committee is indicative of the importance of the subject, and that the contest itself will be watched with marked interest on both sides of the Atlantic. Frank Stockton's new story, "Mrs. Cliff's Yacht," which begins in the April *Cosmopolitan*, promises to be one of the most interesting ever written by that fascinating story-teller. Readers of "The Adventures of Captain Horn" will find in "Mrs. Cliff's Yacht" something that they have been waiting for.


JOHN NEWTON LYLE.

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BIOGRAPHY.

JOHN NEWTON LYLE.

BY F. P. MATZ, SC. D., PH. D., PROFESSOR OF MATHEMATICS AND ASTRONOMY IN IRVING COLLEGE, MECHANICSBURG, PENNSYLVANIA.

JOHN NEWTON LYLE was born in Ralls County, Missouri, March 5, 1836.

"The space in which this county is located is trinally extended, and therefore objective. It has no curvature, either positive or negative. Here planes are flat, and perpendiculars to a transversal are equidistant.

If Lobatschewsky had been born and raised in Ralls County, he would perhaps never have doubted *that two straight lines equidistant from each other may be drawn in the same plane*, nor written a theory of parallels in which this postulate of sound geometry is discredited. The hills, rocks, streams, trees, and hard-pan of Ralls County exist in tridimensional space,—objective to the minds of the inhabitants who till the soil, feed the herds, quarry the rock, fell the trees, and hunt the wild game. These physical objects are realities having an *objective* existence ; and the space occupied by them, and that in which they are contained, is also an objective *entity* although not a *material* one. It is believed that Helmholtz was right in maintaining against Kant the objectivity of space, but wrong in regarding it as a *physical thing* to be moulded like potter's clay. Sir Isaac Newton held the opinion that space was immaterial, immovable, and unalterable, as well as trinally extended, continuous, and unbounded.

Immanuel Kant professed to cognize a real, objective, *extended* world as existing in *a space that*, according to his philosophy *had no existence outside of his own mind.* It would seem that if there is an extended world, there must be

an extended spatial entity to contain it. If space is not *extended*, and therefore not objective, there can be no real, extended, objective, material world.

If Immanuel Kant's experiences in early life had been those of a pioneer's son in Ralls County, Missouri, he would not in all probability have undertaken in his riper years the contract of building a *real world* in a *non-existent* place.

If Fichte, or Hegel, had ever galloped after a wild steer for half a day through a Ralls County forest or been thrown from a bucking mustang on the phenomenal hard-pan of northeastern Missouri, they doubtless would not have felt inclined to regard this real matter-of-fact world as *an idealistic dream*."

The ancestors of John Newton Lyle, on the paternal side, came from northern Ireland ; and on the maternal side, from England and Wales. They settled in Berkeley County, Virginia, in the last century ; and many of their descendants are still to be found there.

Samuel Oldham Lyle, the father of the subject of this sketch, emigrated from Berkeley County, Virginia, in 1832, to Ralls County, Missouri, where he purchased a farm ; married, Ann Rebecca, the daughter of William Gerard, and reared his family. This pioneer couple were intellectual in their tastes, great readers, ambitious to make a pleasant home for their children, and give each of them an education.

Ann Rebecca's father, William Gerard, emigrated from Berkeley County, Virginia, to Kentucky, during the last decade of the last century, where he learned the printing business, and edited and published for many years the *Argus*, a newspaper, at Frankfort, Kentucky. He was a man of affairs, as well as of extensive reading ; and he was also a practical politician intimately associated with the statesmen of his adopted State. He came with his family to northeastern Missouri, in 1830.

John Newton Lyle, in his early boyhood, was thrown in his Grandfather Gerard's society a great deal, and received from him powerful impulses towards intellectual pursuits. The venerable man treated his grandson more as a companion than as a boy needing a rod for his misdemeanors, aroused his curiosity by well-directed questions, corrected his mistakes, and entertained him with anecdotes about Amos Kendall, the elder Blair, and Henry Clay.

Samuel Oldham Lyle was an enterprising, independent, and fearless pioneer, passionately fond of the chase and free life in the wild west ; but, at the same time, he was diligent in his farming and stock-raising. He was a man of quick intelligence, unfailing memory, and sound judgment, who appreciated the value and importance of education ; and he gave to his children the best school advantages that his circumstances would allow.

Young Lyle, at six years of age, was placed in a district school ; and here he remained until he reached the age of twelve. In November, 1848, he entered a classical school taught by the Rev. William T. Dickson, at West Ely, Marion County, three miles from the home of the young pupil. He studied the rudiments of the Latin and Greek languages, Euclid's Elements, and Day's Algebra.

Mr. Dickson was a native of the State of Maine, and came West with Dr.

Ezra Styles Ely, to attend Marion College, some time in the Thirties. "He was an enthusiastic and successful instructor in the branches of learning that he professed to teach. He did not tell his scholars anything about differential coefficients, integrals, or Cartesian co-ordinates. He was silent as to determinants, trilinears, and Non-Euclidean Geometry. He did understand Euclid's Elements, however; and he taught the science, clearly, thoroughly, and ably. With him, straight lines were never *flexed* or *curved*. Tangents to circumferences were never confounded with the curves to which they were tangent. Planes were flat superficies; and, in no instance, were they spherical or pseudo-spherical. He showed the *meaning* of demonstration, by *demonstrating* theorems. He illustrated by practical examples, the difference between direct and indirect proof. He was a true teacher, and succeeded well in imparting to his pupils something of his own appreciation and admiration of the enduring work of Alexandria's immortal geometer."

In the fall of 1851, within three miles of Samuel Oldham Lyle's farm, Van Rensselaer Academy was founded, at the head of which was the Rev. J. P. Finley, afterwards a professor in Westminster College, and the founder of a classical institution at Brookfield, Missouri. John Newton Lyle entered this Academy, in October, 1851; and he was a student there three successive winters, farming during the summer. Here he continued his studies in Latin and Greek, reviewed Euclid, then took up Davies' Legendre and Robinson's Algebra. At this time, he, also, studied Trigonometry and Surveying. Mr. Finley's tastes were classical, rather than mathematical; and his pupil, J. N. Lyle, whilst at the Academy, devoted his energies almost exclusively to mastering the Latin and Greek texts put into his hands.

Before he was nineteen years of age he took charge of his first school in Monroe County, early in September, 1854; thus he began his long career as a teacher, which he has continued almost uninterruptedly until the present time. He worked with the definite plan of preparing for College and earning the funds necessary for securing a collegiate education. He taught two years in the public schools of Monroe County, spending his evenings and Saturdays in study.

During these years he plodded without assistance, through Davies' Analytical Geometry. Having finished this self-imposed task, a strong desire took possession of him to advance farther and investigate Davies' Differential and Integral Calculus. Accordingly one sultry day in August, 1856, he rode from his father's farm to Hannibal, purchased the book, and on returning home immediately sought a secluded spot in the forest and began the study of the first differential coefficient as explained by Charles Davies. He was *thoroughly disgusted* that hot August afternoon, with Davies' description of differentials as the "traces" of vanishing increments. He persevered, however, notwithstanding his dissatisfaction with the author's theory of differentials and differential coefficients. A copy of Loomis's Calculus, which came in his way, was eagerly studied. Loomis's theory of differentials as *rates of variation* had the advantage of being intelligible, and certainly offered something more substantial to be grasped by

the mind than a mere "trace" of a vanishing increment or the "ghost of a departed quantity."

"Rates of variation are *finite* quantities. If differentials are rates of variation, then, of necessity, they must be definite quantities. The Leibnitzian hypothhesis that differentials are *infinitely small quantities* contradicts the hypothesis that they are rates of variation." During the fall of 1856, he studied both Loomis and Davies on the Calculus. This work was done entirely without the instruction of a teacher ; because there was no one within reach, who had studied these branches, to whom he could apply for aid. "*This method of study*, whilst laborious and beset with many inconveniences, *was conducive to independence of thought and action, and the formation of the habit of self-reliance.*"

The first part of the year 1857, John Newton Lyle taught mathematics in Bethel College, a Baptist Institution located at Palmyra, Missouri. The opportunity of attending Marietta College, for which he had long planned and toiled, now presented itself. On examination he entered the Junior Class in Marietta College, the fall of 1857 ; and he continued in that Institution, until his graduation in 1859.

President Israel Ward Andrews conducted the examination in Mathematics, and expressed himself as highly gratified with the candidate's proficiency ; and on making inquiry as to who taught him Analytical Geometry, seemed amused when informed that his only instructor was the youthful pedagogue before him seeking admittance to the privileges of the College. Dr. Andrews was his warm and steadfast friend, from the date of that morning's interview on Mathematics.

J. N. Lyle, in College, sought to utilize the advantages of the library and his literary society as well as those of the recitation-room and the laboratory. His special delight was to participate in the Saturday-morning debates held in the hall of the Alpha Kappa Society. The enjoyableness of the excitement far outweighed the unpleasantness of the collisions incident to such exercises.

He lost no time in obtaining from the College Library De Morgan's Differential and Integral Calculus, in order to learn that author's opinions respecting the principles of the science. "He was interested in noting that De Morgan employed variables that increased, and decreased, indefinitely without limit, instead of the hierarchy of infinitely great, and infinitely small, quantities of the Leibnitzian hypothesis. Whilst no lost value was attributed to these variables, every value that they did have, was *finite*. The hypothesis of increasing, and decreasing, variables having finite values not only works well in practice, but has the advantage over the hypothesis of Leibnitz in that it is intelligible and does not involve contradiction. It also harmonizes well with the view that differentials are rates of variation. Further, in considering a limit, we note that the interval between a limit and the variable that approaches it, is itself a variable that decreases without limit. From this point of view, the absurdity of regarding a variable that increases without limit as having a limit appropriately symbolized by ∞, is quite evident.

No benefit accrues to the Science of the Calculus, from De Morgan's hypothesis that there are *two* kinds of zeros—*the absolute zero,* and *the indifinitely small quotient.* The absolute zero is destitute of all value ; in fact, it is the negation of quantity,—and hence can not be treated as quantity, without violating the logical law of Non-Contradiction. A quotient may become indefinitely small, but can not become so small as not to be quantity. To name a quotient zero, is manifestly a misnomer. Mathematical and logical confusion is liable to result from the ambiguous use of the symbol 0. Treating quantity as no quantity, or no quantity as quantity, is a procedure which may be profitably dispensed with in Mathematics."

E. W. Evans, of Yale, came to Marietta College, as Professor of Mathematics, at the same time that J. N. Lyle entered as a student. The young professor seemed very lonesome as his wife remained in the East that fall. He would come over to Lyle's room of evenings and remain for hours. His conversation which took a wide range was quite instructive to his western pupil. Mathematics was discussed a great deal, but not exclusively. He believed most religiously that "brevity is the soul of wit." He once said : "Lyle, the longer I live the more I like 'short things'." His pupil furnished his share of the intellectual picnic with anecdotes and experiences respecting that portion of the West where Mark Twain was born, Tom Sawyer flourished, and Captain Sellers bored with a big auger.

The two years immediately after graduation he spent in teaching in Pettis and Morgan Counties, Missouri. His leisure hours were occupied in reading law-books. In the spring of 1862, he was offered the chair of Mathematics and Natural Science in Westminster College, a position he held until 1865, when he went to Carondelet, a suburb of St. Louis, where he taught a Grammar School ; but in the fall of the same year, he accepted the position of Acting Professor of Mathematics and Natural Science in his *Alma Mater*, Marietta College. He continued there three years ; at the expiration of which time, he returned to Fulton, as Professor of Natural Science in Westminster College. Here he has since remained. First and last, as the exigencies of College-work might require, he has taught branches in nearly every department of the Institution.

He is an active member of *The Missouri Teacher's Academy.* To educational journals he has contributed hundreds of articles principally on Mathematical Philosophy. During the last three years preceding 1890, he had published in the *Missouri School Journal* not less than sixty-one articles. He has written an unpublished manuscript on the Differential and Integral Calculus.

The degree of Ph. D., was conferred on him by Marietta College, in 1881. In 1868, Professor Lyle was married to Miss Margaret T. Hays, daughter of John B. Hays, M. D., of Marion County, Missouri, who until her death, December 26, 1882, in spite of ill health and great suffering, led such a life of unselfish devotion to husband, children, and friends, as called forth constant admiration of the talent, energy, and piety, that enabled her to accomplish so much. Three of the five children of this couple are living, two daughters and a son, Rev. Edward

Hays Lyle, an alumnus of Westminster College, a Theological student of Princeton Seminary for two years, and at present a minister in charge of a church at La Junta, Colorado.

Dr. Lyle, in 1884, married his second wife, Miss Mattie E. Grant, a scholarly and cultured lady, of Bardstown, Kentucky.

Dr. Lyle has been for many years an Elder in the Presbyterian Church, the church of his ancestors for, at least, the century and a half that have elapsed since his Great Grandfather emigrated from the northern part of Ireland to Berkeley County, Virginia.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Concluded.]

We will now find the centroid of the eighth part of the surface

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1, \text{ I, when } c=b, \text{ II, } c=a.$$

We have $\bar{x}=\dfrac{\int xds}{\int ds}$, $\bar{y}=\dfrac{\int yds}{\int ds}$, $\bar{z}=\dfrac{\int zds}{\int ds}$.

I. $$s=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}dxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\tfrac{1}{4}\pi b\left(b+\frac{a}{e}\sin^{-1}e\right).$$

$$s.\bar{x}=\int xds=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}xdxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,xdx=\frac{\pi ab(a^2+ab+b^2)}{6(a+b)}.$$

$$\therefore\ \bar{x}=\frac{2a(a^2+ab+b^2)}{3(a+b)(b+\frac{a}{e}\sin^{-1}e)}.$$

$$s.\bar{y}=s.\bar{z}=\int yds=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}ydxdy$$

$$=\frac{b^2}{a^2}\int_0^a\sqrt{(a^2-x^2)(a^2-e^2x^2)}\,dx=ab^2\int_0^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\cos^2\theta d\theta,\ x=a\sin\theta.$$

$$=\frac{ab^2}{3e^2}\left\{(1+e^2)\,E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}.$$

$$\therefore\ \bar{y}=\bar{z}=\frac{4ab\left\{(1+e^2)E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}}{3\pi e^2(b+\frac{a}{e}\sin^{-1}e)}.$$

II. $$s=\frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}dydx$$

$$=\frac{\pi a}{2b^2}\int_0^b\sqrt{b^4+(a^2-b^2)y^2}\,dy=\frac{\pi a^2}{4}\left\{1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}.$$

$$s.\bar{x}=s.\bar{z}=\int xds=\frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}xdydx.$$

$$s.\bar{x}=s.\bar{z}=\frac{a^2}{b^3}\int_0^b\sqrt{(b^2-y^2)(b^4+a^2e^2y^2)}\,dy$$

$$=a^2\int_0^{\frac{1}{2}\pi}\sqrt{b^2+a^2e^2\cos^2\theta}\sin^2\theta d\theta,\ y=b\cos\theta$$

$$=a^3\int_0^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\sin^2\theta d\theta$$

$$= \frac{a^3}{3e^2}\left\{(1-e^2)F(e, \frac{\pi}{2})-(1-2e^2)E(e, \frac{\pi}{2})\right\}.$$

$$\therefore \bar{x}=\bar{z}=\frac{4a\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}}{3\pi e^2(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

$$s.\bar{y}=\frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}ydydx=\int yds$$

$$=\frac{\pi a}{2b^2}\int_0^b\sqrt{b^4+a^2e^2y^2}\,ydy=\frac{\pi ab(a^2+ab+b^2)}{6(a+b)}.$$

$$\therefore \bar{y}=\frac{2b(a^2+ab+b^2)}{3a(a+b)(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

Since the limit of $\frac{\sin^{-1}e}{e}$ and $\frac{\log\frac{1+e}{1-e}}{2e}$ is 1 when $e=0$ we have, in either case, when $a=b$, $\bar{x}=\bar{y}=\bar{z}=\frac{1}{2}a$. The surface of the fourth part of the paraboloid $x^2+y^2=2a^2z$, for $z=h$.

$$s=\iint\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dzdx=\iint\sqrt{1+\frac{x^2}{y^2}+\frac{a^4}{y^2}}\,dxdz.$$

$$\therefore s=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\,dzdx=\frac{\pi a}{2}\int_0^h\sqrt{a^2+2z}\,dz$$

$$=\frac{\pi a}{6}\left\{(a^2+2h)^{\frac{3}{2}}-a^3\right\}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{a^2+2z}\,dzdx=a^2\int_0^h\sqrt{(a^2+2z)2z}\,dz$$

$$=\frac{a^2}{16}\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}.$$

$$\therefore\ \bar{x}=\bar{y}=\frac{3a\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}}{8\pi\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

$$s.\bar{z}=\int zds=a\int_o^h\int_o^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\,zdzdx$$

$$=\frac{\pi a}{2}\int_o^h\sqrt{a^2+2z}\,zdz=\frac{\pi a}{30}\left\{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5\right\}.$$

$$\therefore\bar{z}=\frac{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5}{5\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

The surface of the fourth part of the cone $x^2+y^2=a^2z^2$, for $z=h$.

$$s=\int\int\sqrt{1+\frac{x^2}{y^2}+\frac{a^4z^2}{y^2}}\,dzdx=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{zdzdx}{\sqrt{a^2z^2-x^2}}$$

$$=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h zdz=\frac{\pi ah^2\sqrt{1+a^2}}{4}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\sqrt{1+a^2}\int_o^h\int_o^{az}zdzdx=a^2\sqrt{1+a^2}\int_o^h z^2dx=\frac{a^2h^3\sqrt{1+a^2}}{3}.$$

$$\therefore\ \bar{x}=\bar{y}=\frac{4ah}{3\pi}.$$

$$s.\bar{z}=\int zds=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{z^2dzdx}{\sqrt{a^2z^2-x^2}}=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h z^2dz=\frac{\pi ah^3\sqrt{1+a^2}}{6}.$$

$$\therefore\ \bar{z}=\frac{2h}{3}.$$

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from March Number.]

Having disposed of the regular primitive groups we turn now to those whose order exceeds their degree. We have proved that all of these involve n conjugate subgroups whose degree is at most equal to $n-1$. Suppose the degree of these subgroups were $n-2$. Without loss of generality we may then assume that the following identities are satisfied :

$$G_1 \equiv G_2, \ G_3 \equiv G_4, \ \ldots\ldots \ G_{n-1} \equiv G_n.$$

If g_1 represents the order of G_1 we see that $2g_1$ substitutions of G transform G_1 into itself, viz., those which replace a_1 by itself and those which replace a_1 by a_2. All of the g_1 substitutions which replace a_1 by a_2 must therefore also replace a_2 by a_1, *i. e.*, contain the cycle $a_1 a_2$. Similar remarks apply to the other couples $a_3 a_4$, ...$a_{n-1} a_n$.

We inquire whether these couples may be used as systems of non-primitivity. We have already proved that every substitution that replaces one letter of a couple by the other contains the couple as a distinct cycle. It remains to show that the couples are interchanged as units by the substitutions of G. Suppose one of these substitutions t replaces a_1 by a_3. Then will

$$t G_3 t^{-1} = G_1$$

t must therefore replace a_2 by a_4. Since similar remarks apply to the other couples we have proved that the couples can be used as systems of non-primitivity.

In an exactly similar way we can prove the general case that if the degree of the conjugate subgroups is $n-\alpha$ $(\alpha>1)$ then systems of α letters each may be used as systems of non-primitivity. Hence the

THEOREM. *Whenever a transitive group contains a subgroup whose degree is less than $n-1$ and which involves all the substitutions that do not contain a given letter it must be non-primitive.*

Having developed some of the most important properties of the subgroups of G which do not contain a given letter we proceed to inquire into their substitutions. Suppose that among the substitutions of a transitive group G

$$a_1 a_\alpha$$

has only one solution ; *i. e.*, there is only one cycle of this type in the group which contains a_1. Then there can be only one value of γ for each β in

$$a_\beta\, a_\gamma \qquad (\beta = 1,\ 2,\ \ldots\ldots n)$$

since any a can be transformed into a_1. All the conjugates of $a_1 a_\alpha$ are therefore distinct and may be used as systems of non-primitivity of the given transitive group.

More generally speaking we may say that if G contains a subgroup G' whose degree n' is less than the degree of G and if any given letter of G (a_1) is found in only one of the transforms of G' with respect to G, then will these transforms

$$G',\ G'',\ \ldots\ldots,\ G^n$$

constitute systems of non-primitivity of G.

For if G^α and G^β had a common letter then would the substitution of G which transforms this common letter into a_1 lead to two such groups both of which would involve a_1. This is contrary to the hypothesis. These conjugate subgroups must therefore involve distinct sets of letters which may be regarded the systems of non-primitivity of G. Hence the

THEOREM. *If a primitive group contains a subgroup whose degree is less than the degree of the group it must also contain a substitution which transforms this subgroup into one which contains any one of its letters together with at least one new letter.*

From this theorem it follows that if a primitive group whose degree exceeds 2 contains the cycle $a_1 a_2$ it must also contain $a_1 a_3$ (a_3 representing any suitable letter, different from a_1 and a_2) and therefore the symmetric group of these three letters $(a_1 a_2 a_3)$all.

If a primitive group whose degree exceeds three contains $(a_1 a_2 a_3)$all it must, according to the given theorem, also contain $(a_1 a_\alpha\, a_\beta)$all where at least one of the two subscripts α, β exceeds 3. Representing this by 4 we can easily show that the group must contain at least all the substitutions of

$$(a_1 a_2 a_3 a_4)\text{all}$$

whose degree does not exceed 3. For if any such substitution is given we can find some substitution of $(a_1 a_2 a_3)$all which is either the same or differs from it only in having another letter a_α where the given substitution has a_4. The transform of this substitution with respect to $a_\alpha\, a_4$ (which is known to be in the group) will be the given substitution. Since every substitution of the fourth degree is the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 a_3 a_4)\text{all}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{all}$$

where the number of subscripts 1, α, ...μ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G must contain

$$a_\alpha\, a_{m+1} \qquad (\alpha = 1, 2, \ldots\ldots a_m).$$

We consider now any substitution of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}$$

whose degree does not exceed m. We can find some substitutions in

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

which is either the same or differs from it only in having a_α where this has a_{m+1}. In this case the transform with respect to $a_\alpha\, a_{m+1}$ will be the given substitution. Since a substitution of the $m+1$ degree ($m \equiv 2$) may be regarded as the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}.$$

Calling $m+1$ m' we can prove in the same way that G contains the symmetric group of $m'+1=m+2$ letters, etc. Hence the

THEOREM. *Whenever a primitive group contains a symmetric subgroup of a lower degree it must be a symmetric group.*

COROLLARY. *If a primitive group contains a substitution of the form* $a_1 a_2$ *it is symmetric.*

We will now suppose that the primitive group contains

$$a_1 a_2 a_3.$$

If its degree exceeds 3 it must also contain

$$a_1 a_\alpha\, a_\beta$$

where at least one of the two letters, say α, is greater than 3. We shall represent this by 4, G then contains the two substitutions

$$a_1 a_2 a_3 \text{ and } a_1 a_4 a_\beta$$

and therefore

$$(a_1 a_2 a_3 a_4)\text{pos}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{pos}$$

where the number of subscripts 1, α, ...μ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G contains

$$a_\alpha\, a_{m+1} a_\beta \qquad (\alpha=1, 2, \ldots\ldots m).$$

It must therefore contain at least all of the substitutions of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{pos}$$

whose degree does not exceed m. For if s is any such substitution containing a_{m+1} there is some substitution s_1 in

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

which differs from s only in having a_δ where s has a_{m+1}. If β exceeds m we make $\alpha=\delta$ then will $a_\alpha\, a_{m+1} a_\beta$ transform s_1 into s. If $\beta \overline{\gtrless} m$ we transform the substitution

$$a_\alpha\, a_{m+1} a_\beta$$

with respect to some substitution of $(a_1 a_2 \ldots\ldots a_m)$pos. So that in place of a_β we may have a letter not found in s. Let this transform be

$$a_\gamma\, a_{m+1} a_\varepsilon \qquad (\gamma,\ \varepsilon \overline{\gtrless} m).$$

We now take from the substitutions of $(a_1 a_2 \ldots\ldots a_m)$pos the one s_2 which differs from s only in having a_γ where s has a_{m+1} if s does not contain a_γ, and the one s_3 which differs from s only in having a_ε, a_γ where s has a_γ, a_{m+1} if s contains a_γ. The transform of these with respect to

$$a_\gamma\, a_{m+1} a_\varepsilon$$

will be the required substitution s.

This proves that G contains all the substitutions of $(a_1 a_2 \ldots\ldots a_{m+1})$pos whose degree is equal to or less than m. These generate $(a_1 a_2 \ldots\ldots a_{m+1})$pos, for any positive substitution of the $(m+1)^{th}$ degree $(m>2)$ may be considered as the product of two positive substitutions of a lower degree. [Let $s=\ldots a_x a_y \ldots$ be any positive substitution of the $(m+1)^{th}$ degree and $s_1 \ldots\ldots a_x a_y \ldots\ldots$ be any positive substitution of a lower than the $(m+1)^{th}$ degree. Then will s_2 in

$$s=s_1 s_2 \text{ or } s_2=s_1^{-1}s$$

be also a positive substitution whose degree $\overline{\gtrless} m$].* Hence the

THEOREM. *If a primitive group contains a substitution of the form $a_1a_2a_3$ but none of the form a_1a_2 it is the alternating group.*

We are now in possession of the following important facts in regard to any primitive group G.

(1) If $g=n$, G must be generated by a single cycle which involves a prime number of letters, and for each prime number there is one and only one such primitive group.

(2) If g does not equal n it must be a larger multiple of n and G must contain n conjugate subgroups whose degree is $n-1$ and whose order is $g\div n$.

(3) If G contains a substitution of the form a_1a_2 or one of the form $a_1a_2a_3$ it must contain the alternating group.

(4) Both the alternating and the symmetric groups have a 1, 1 correspondence to the positive integers beginning with 2.

(5) The order of the symmetric group is $n\,!$ and that of the alternating group is $\frac{1}{2}n\,!$.

(6) The average number of letters in all the substitutions of a transitive group is $n-1$.

(7) Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

The three classes of primitive groups, regular, alternating, and symmetric, each of which contains an infinite number of members, are distinct when $n>3$. The groups that belong to these classes for any value of n are well known. It remains to determine those whose order satisfies the inequality

$$n>g>\tfrac{1}{2}n\,!.$$

Before pursuing the general discussion any farther we shall seek all the primitive groups whose degree does not exceed six. In doing this we shall use some methods which will be of service in the further study of this subject. Most of the methods, however, may serve as illustrations of the theorems which have been developed.

[To be Continued.]

*It can be easily proved that if a group contains

$$a_1a_2a_\alpha \qquad (\alpha=1,\ 2,\ \ldots\ldots n)$$

it contains the alternating group of degree n, and if it contains

$$a_1a_\alpha \qquad (\alpha=1,\ 2,\ \ldots\ldots n)$$

it contains the symmetric group of degree n. Cole's Netto, §§ 34, 35.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

Corollary II. But again I am able hence to show that those two straights AX, BX, meeting which the straight $PFHD$ makes either two internal angles toward the same parts equal to two right angles, or consequently (from Eu. I. 13 and 15) alternate external or internal angles equal to one another, or again, from the same cause, an external (as suppose DHX) equal to an internal and opposite HFX; that, say I, those two straights not even in their infinite production can meet one another.

For if from any point N of AX is let fall to BX the perpendicular NR, this will be in the hypothesis of acute angle (which alone in any case can hinder us) greater (from III. Cor. I.) than the common perpendicular KL. Therefore those two straights AX, BX cannot ever meet one another.

But furthermore here thou hast demonstrated propositions 27 and 28 of the first book of Euclid, and indeed without immediate dependence from the preceding 16 and 17 of the same first book, about which difficulties could arise when the triangle should be of infinite sides on a finite base; to which sort of a triangle without doubt would refer one who believed that these two straights AX, BX met one another at least at an infinite distance, although the angles at the transversal $PFHD$ were such as we have supposed.

Moreover, on account of the demonstrated common perpendicular KL, surely those two KX, LX cannot come together toward the part of the points X, since also (from a superposition easily understood) toward the other part also would meet at the same time the remaining and themselves unterminated KA, LB. Wherefore two straights AX, BX would enclose a space; which is contrary to the nature of the straight line.

But these things are later. For in the preceding I have never applied either the 16th or 17th of the first book of Euclid, except where clearly it treats of a triangle bounded on every side, as indeed I promised I would so take care to do *in Proemio ad Lectorem*.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

V. Let ABC be $\triangle$ right-angled at C. Draw FD perpendicular to AB, meeting either leg produced. There are thus four similar right triangles.

Fig. 5.

Letting $AC=b$, $AB=c$, $BC=A$, $CD=x$, $CE=y$, $AF=z$, $EB=a-y$, $FB=c-z$, $AD=b+x$, $FE=v$, $ED=w$, $FD=v+w$, we obtain the following proportions, with their resulting equations :

(1). $b:z::c:b+x$. $\therefore b(b+x)=cz$........1.

(2). $b:z::a:v+w$. $\therefore b(v+w)=az$......2.

(3). $c:b+x::a:v+w$. $\therefore c(v+w)=a(b+x)$....................3.

(4). $b:y::c:w$. $\therefore bw=cy$......................................4.

(5). $b:y::a:x$. $\therefore bx=ay$......................................5.

(6). $c:w::a:x$. $\therefore cx=aw$......................................6.

(7). $b:v::c:a-y$. $\therefore b(a-y)=cv$.........................7.

(8). $b:v::a:c-z$. $\therefore b(c-z)=av$.........................8.

(9). $c:a-y::a:c-z$. $\therefore c(c-z)=a(a-y)$.....................9.

(10). $z:y::b+x:w$. $\therefore zw=y(b+x)$.........................10.

(11). $z:y::v+w:x$. $\therefore xz=y(v+w)$.........................11.

(12). $b+x:w::v+w:x$. $\therefore x(b+x)=w(v+w)$................12.

(13). $z:v::b+x:a-y$. $\therefore z(a-y)=v(b+x)$..................13.

(14). $z:v::v+w:c-z$. $\therefore z(c-z)=v(v+w)$..................14.

(15). $b+x:a-y::v+w:c-z$. $\therefore (c-z)(b+x)=(a-y)(v+w)$....15.

(16). $y:v::w:a-y$. $\therefore y(a-y)=vw$......................... 16.

(17). $y:v::x:c-z$. $\therefore y(c-z)=vx$.........................17.

(18). $w:a-y::x:c-z$. $\therefore w(c-z)=x(a-y)$..................18.

We are now to find combinations of the above equations from which the elements x, y, z, v, w, can be eliminated, thus leaving us the relation existing between a, b, and c.

It is evident that from no single equation, nor from any set of two equations, can the relation be determined.

There remains three possible cases of combinations to be considered :

1. When three of the elements x, y, z, v, w, are involved.
2. When four.
3. When five, or all.

First Case. Of this case there are $\frac{5.4.3}{|\underline{3}}=10$ possible combinations of three unknown elements : v, w, x ; v, w, y ; and so on.

Before taking up these in detail, we note that by inspection of the proportions, it easily may be seen that the following eighteen sets of equations each comprise dependent equations :

1, 2, 3 ; 4, 5, 6 ; 7, 8, 9 ; 10, 11, 12 ; 13, 14, 15 ; 16, 17, 18 ; 1, 4, 10 ; 1, 7, 13 ; 2, 5, 11 ; 2, 8, 14 ; 3, 6, 12 ; 3, 9, 15 ; 4, 7, 16 ; 5, 18, 17 ; 6, 9, 18 ; 10, 13, 16 ; 11, 14, 17 ; 12, 15, 18.

Hence, in our search for possible combinations, all such must be rejected as contain any of these sets.

There are three equations involving v, w, x: 3, 6, 12. But this combination must be rejected for the reason just given. For the same reason, or because there is wanting a sufficient number of equations involving the three unknown elements, the other nine combinations must be rejected, except the combination x, y, z, which elements are involved in equations 1, 5, 9. If we eliminate x, y, z from these equations, we obtain the desired relation, $c^2=a^2+b^2$.

It should be observed, in passing, that future combinations including 1, 5, 9, must also be rejected.

Second Case. Of this case there are $\frac{5.4.3.2}{|\underline{4}}=5$ possible combinations of four unknown elements; and, besides, the exceptional combination, $v+w$, x, y, z, $v+w$ being regarded as a single unknown.

Before proceeding to investigate this case, it is necessary to call attention to sets of four dependent equations. Take, for example, the set 1, 2, 6, 12. From 1 and 2, 3 is obtained. But 3 with 6 and 12 gives a set of three dependent equations ; hence the set 1, 2, 6, 12 must be rejected. A little stndy of the eighteen sets given in Case 1, will disclose forty-five sets of four dependent equations.

The equations involving the unknown elements v, w, x, y, are 3, 4, 5, 6, 7, 12, 16. Out of these seven equations, there are $\frac{7.6.5.4}{|\underline{4}} = 35$ combinations, taking four at a time. Of these thirty-five sets, fourteen are to be rejected, for reasons previously stated. The remaining twenty-one sets, of which 7, 5, 4, 3, is a type, and to which the other twenty easily can be reduced, give, after the unknown elements have been eliminated, the desired relation between a, b, and c.

Similarly, we find twenty-one sets each of four equations, involving (v, w, x, z) and (v, w, y, z), and seventeen each involving (v, x, y, z), (w, x, y, z),

and $(v+w, x, y, z)$, thus making in all 114 proofs for this case.

THIRD CASE. Of this case, there are $\frac{18.17.16.15.14}{\lfloor 5}=8568$ sets of the eighteen equations, taking five at a time.

To determine how many of this number must be rejected, proceed as follows. Begin with the list of sets of dependent equations found in Case 1. Notice that there are $\frac{15.14}{\lfloor 2}=105$ sets of the eighteen equations taking five at a time, each containing equations 1, 2, 3; the same number containing equations 4, 5, 6; and so on, till we come to 1, 4, 10; for while there are 105 sets containing equations 1, 4, 10, three of them have already been counted out. So proceed, with the entire list of sets of dependent equations in Case 1, then with the set 1, 5, 9, following this with the sets of Case 2. We thus find that there are 3746 sets of five to be rejected, either because they contain sub-sets of dependent equations or sub-sets of equations from which the desired relation between a, b, c, is obtained.

One more class must be rejected: sets of five dependent equations. For example, 10, 9, 7, 6, 3, which is a type of all the others—72 in number—and from which the 72 can easily be deduced.

Deducting from 8568, 3746+73, we have remaining 4749 sets of five, from which can be derived the identity $c^2=a^2+b^2$.

$\therefore$ $1+114+4749=4864$, the number of proofs by this method.

EXAMPLES:

1. $cv+cw-ax=ab$ .... 3.
 $bw=cy$ .... 4.
 $aw=cx$ .... 6.
 $cv+by=ab$ .... 7.
 4 in 6, $bx=ay$ .... 5.
 4, 5 and 7 in 3, $ab-by+\frac{c^2y}{b}-\frac{a^2y}{b}=ab$. $\therefore$ $c^2=a^2+b^2$.

2. $cz-bx=b^2$ .... 1.
 $bv+bw-az=0$ .... 2.
 $bw=cy$ .... 4.
 $bx=ay$ .... 5.
 $cv+by=ab$ .... 7.
 1 in 2, $cv+cw-ax=ab$ .... 3.
 4, 5, and 7 in 3, same as in 1st example.

VI. Let ABC be $\triangle$ right-angled at C. Produce AC to some point as D. Draw DF perpendicular to AB, produced, and meeting CB, produced.

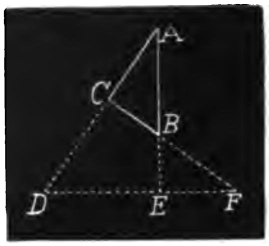


Fig. 6.

Employing notation similar to that used in V., and proceeding somewhat in the same manner, we find that this method also yields a large number of proofs, in fact the same number that we found in V.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Prophsed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of $a=3$, $b=4$, and $c=5$ miles, per hour. They start from Washington, at $m=1$, $n=2$, and $p=3$ o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

Solution by P. S. BERG, Larimore, North Dakota.

Since B gains 1 mile in 1 hour on A, to gain 3 miles will require 3 hours, or it will be 5 o'clock and 12 miles from starting point when B and A meet. C has traveled 10 miles. Since B and C travel 9 miles in 1 hour, they will travel 2 miles in $\frac{2}{9}$ hour, hence they will meet at $5\frac{2}{9}$ o'clock. Since A and C travel 8 miles in 1 hour, they will travel 2 miles in $\frac{1}{4}$ hour, hence they will meet at $5\frac{1}{4}$ o'clock.

In case all three continue walking ahead, as stated above A and B will meet at 5 o'clock. Since C gains 2 miles on A in 1 hour, to gain 6 miles will require 3 hours. Hence they will meet at 6 o'clock. Since C gains 1 mile on B in 1 hour, to gain 4 miles will require 4 hours. Hence it will be 7 o'clock when they meet.

Also solved by *B. F. YANNEY* and *H. C. WILKS*.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{13}{28}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt, and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithimetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor*.)

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

1. By first condition, the eating capacity of a colt is to that of an ox as 6 : 10.

$\therefore$ By last condition, the eating capacity of a colt is to that of a sheep as 589 : 21.

$\therefore$ The eating capacity of a colt is to that of a sheep, an ox, and a colt together, as 1767 : 4775.

2. $\therefore$ The first two conditions of the problem may be stated as follows:

10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 17 colts could eat up 6 acres, or 1400 colts would require $2\frac{6}{7}$ weeks longer than 1540 colts to eat up 589 acres.

3. Let $42u$ be the amount of grass consumed each week by a colt.

4. Suppose the time it takes 10 colts to eat up 3 acres is 18 weeks; then, the time it takes 17 colts to eat up 6 acres would be 25 weeks.

5. $\therefore$ $(10\times18\times42u)\div3=2520u$, total amount of pasture eaten from 1 acre in 18 weeks; and $(17\times25\times42u)\div6=2975u$, total amount of pasture eaten from 1 acre in 25 weeks.

6. $\therefore$ $(2975u-2520u)\div(25-18)=65u$, obviously the amount of growth on 1 acre in 1 week, and the same result that would be obtained whatever the time supposed in (4).

7. $\therefore$ $2520u-18\times65u=1350u$, amount of pasture originally on 1 acre.

8. $\therefore$ $(589\times1350u)\div(1400\times42u-589\times65u)=\frac{159030}{4103}$, the number of weeks it would take 1400 colts to eat of 589 acres of pasture; similarly, the time required for 1540 colts is found to be $\frac{159030}{5279}$ weeks. Now, the difference between these two numbers, $\frac{159030\times1176}{4103\times5279}$ weeks : $2\frac{6}{7}$ weeks, the true difference :: 18 weeks, the supposed time : the true time.

9. $\therefore$ Since the only number that needs correcting, to enable us to complete the solution, is $1350u$, the amount of pasture originally on 1 acre, the time required for an ox, a colt, and a sheep together to eat up 1 acre, is

$$\left(\frac{20}{7}\times\frac{4103\times5279}{159030\times1176}\times1350u\right)\div\left(\frac{4775}{1767}\times42u-65u\right)=9\frac{2482499}{11757354}\text{weeks. Answer.}$$

H. C. Wilkes gets 142.25+ days.

NOTE. This problem appeared a few years ago in the *School Visitor*. With no little difficulty, we obtained a solution by Algebra. The solution was not published because of the difficult composition. It is strange that such a problem should appear in an arithmetic which is to be used by boys and girls 16 years old and upwards. EDITOR.

PROBLEMS.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for $25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic.*]

59. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me $1\frac{1}{2}$ per cent. brokerage for buying some uncurrent bank bills at 20 per cent. discount. Of these bills 4 of $50. each become worthless, but the remainder I dispose of at par, and make by the operation $364. What was the face amount? [Which answer is correct, $3000, or $$3048\frac{24}{37}$?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

(a) How much can be paid for a bond, bearing 5 per cent. interest, and having ten years to run, so as to realize 3 per cent. on the investment? (b) At what price must the government sell 5 per cent. $100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3 per cent. bonds at par, to run ten years, interest payable annually, provided the buyer can invest all interest received at 4 per cent. interest, payable annually?

Solution by J. K. ELWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Let x=price, a=face, n=number of periods, R=rate bond bears, r=rate to be realized, r'=rate on interest.

The interest on bond is an annuity at compound interest whose final value $=\frac{Ra}{r'}[(1+r')^n-1]$, which added to the *face* value of bond must equal the compound amount of the *price* for n periods, or $x(1+r)^n$.

$\therefore x=\dfrac{a+Ra[(1+r')^n-1]}{(1+r)^n}$. For ($a$), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.03$.

$$\therefore\ x=\frac{100+\frac{.05}{.03}(1.03^{10}-1)}{1.03^{10}}=\$117.0604.$$

For (b), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.04$.

$$\therefore\ x=\frac{100+\frac{.05}{.04}(1.04^{10}-1)}{1.03^{10}}=\$119.0777.$$

If in (a) interest were payable semi-annually, we should have $a=100$, $n=20$, $R=.025$, $r=.015$, $r'=.015$, and $x=\$117.168+$, or $\$117.17$ as given in the tables of bond values used by brokers and bankers.

Also solved by *E. W. MORRELL*, *B. F. YANCEY* and *G. B. M. ZERR*. Prof. Morrell obtained as results $118.856 and $117.661; and Proposer, to last part, $117.60.

57. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2\ldots\ldots a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2\ldots\ldots a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2\ldots\ldots a_3^2 \\ \cdots & \cdots & \cdots \\ a_n^2 & a_n^2 & a_n^2\ldots s-a_n^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1\ldots\ldots\ldots\ldots a_1 \\ a_2 & s-a_2 & a_2\ldots\ldots\ldots\ldots a_2 \\ a_3 & a_3 & s-a_3\ldots\ldots\ldots a_3 \\ \cdots & \cdots & \cdots \\ a_n & a_n & a_n\ldots\ldots\ldots s-a_n \end{vmatrix}$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let Q=the quotient and as we can exchange row for column without altering the value, we get

$$Q=\begin{vmatrix} (s-a_1)^2 & a_2^2 & a_3^2\ldots\ldots a_n^2 \\ a_1^2 & (s-a_2)^2 & a_3^2\ldots\ldots a_n^2 \\ a_1^2 & a_2^2 & (s-a_3)^2\ldots\ldots a_n^2 \\ \cdots & \cdots & \cdots \\ a_1^2 & a_2^2 & a_3^2.(s-a_n)^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_2 & a_3\ldots\ldots\ldots a_n \\ a_1 & s-a_2 & a_3\ldots\ldots\ldots a_n \\ a_1 & a_2 & s-a_3\ldots\ldots\ldots a_n \\ \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3\ldots\ldots s-a_n \end{vmatrix}$$

All the elements in the i^{th} column of the numerator being a_i^2, of the denominator a_i, except in the i^{th} row which is $(s-a_i)^2$ for numerator, and $s-a_i$ for denominator. Hence, we have

$$Q=\begin{vmatrix} 1, & 0, & 0, & 0,\ldots\ldots \\ 1, & (s-a_1)^2, & a_2^2, & a_3^2,\ \ldots\ldots \\ 1, & a_1^2, & (s-a_2^2,) & a_3^2,\ \ldots\ldots \\ 1, & a_1^2, & a_2^2, & (s-a_3)^2,\ \ldots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \div \begin{vmatrix} 1, & 0, & 0, & 0,\ldots\ldots \\ 1, & s-a_1, & a_2, & a_3,\ldots\ldots \\ 1, & a_1, & s-a_2, & a_3,\ldots\ldots \\ 1, & a_1, & a_2, & s-a_3,\ \ldots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Multiply first column of numerator by a_i^2, of the denominator by a_i and subtract from the i^{th} column; do this for each column and the value is unaltered.

$$\therefore Q=\begin{vmatrix}1, & -a_1^2, & -a_2^2, & -a_3^2, \ldots \\ 1, & s(s-2a_1), & 0, & 0, \ldots \\ 1, & 0, & s(s-2a_2), & 0, \ldots \\ 1, & 0, & 0, & s(s-2a_3), \ldots \\ \ldots & \ldots & \ldots & \ldots\end{vmatrix} \div \begin{vmatrix}1, & -a_1, & -a_2, & -a_3, \ldots \\ 1, & s-2a_1, & 0, & 0, \ldots \\ 1, & 0, & s-2a_2, & 0, \ldots \\ 1, & 0, & 0, & s-2a_3, \ldots \\ \ldots & \ldots & \ldots & \ldots\end{vmatrix}$$

Let $u=(s-2a_1)(s-2a_2)(s-2a_3)\ldots\ldots(s-2a_n)$.

$$\sum\frac{a_i^2}{s-2a_i}=\frac{a_1^2}{s-2a_1}+\frac{a_2^2}{s-2a_2}+\frac{a_3^2}{s-2a_3}+\ldots\ldots$$

$$\therefore Q=\frac{s^{n-1}u\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{u\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}=\frac{s^{n-1}\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}.$$

ERRATA. On page 52 of last issue, line 3 from bottom, read = before $\frac{1}{c}$, and in the denominator read $\sqrt{a^2-x^2}$ for "$\sqrt{a^2+x^2}$"; on page 53, line 15, extend the radical sign over a^2-x^2 and b^2-x^2, in the numerators.

PROBLEMS.

64. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2}\ldots\ldots(1).$$
$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2}\ldots\ldots(2).$$

65. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* or *even*.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

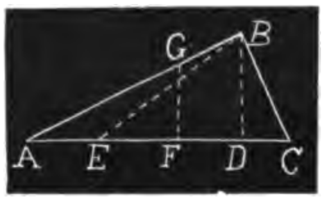
SOLUTIONS OF PROBLEMS.

50. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Divide a triangle into the ratio of m to n by a line perpendicular to the base.

Solution by J. C. GREGG, Superintendent of Schools, Brazil, Indiana; E. W. MORRELL, Professor of Mathematics, Montpelier Seminary, Montpelier, Vermont; and the PROPOSER.

Let ABC be the triangle. Draw the altitude BD. Divide the base AC at E so that $AE:EC=m:n$. Draw the line BE.



Then $\triangle ABE : \triangle EBC = AE : EC = m : n$.....(1).

Take AF a mean proportional between AE and AD, then draw GF parallel to BD.

Then $\triangle AFG : \triangle ADB = AF^2 : AD^2$.

But $AF^2 = AE \times AD$.

$\therefore \triangle AFG : \triangle ADB = AE \times AD : AD^2 = AE : AD = \triangle ABE : \triangle ADB$.

$\therefore \triangle AFG = \triangle ABE$ and $\triangle EBC = FGBC$.

Hence, using in (1), we have $\triangle AFG : FGBC = m : n$. Q. E. D.

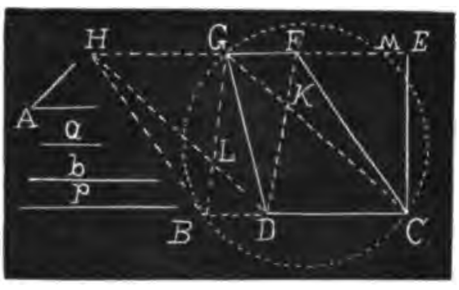
Also solved in various ways by *G. B. M. ZERR, B. F. YANNEY, J. SCHEFFER, A. H. BELL, F. R. HONEY, O. W. ANTHONY, H. J. GAERTNER, G. I. HOPKINS, J. M. COLAW, J. O. MAHONEY.*

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Construct a tropezoid, given the bases, the altitude, and the angle formed by the intersection of the diagonals.

Solution by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; FREDERICK R. HONEY, A. B., New Haven, Connecticut; J. SCHEFFER, Hagerstown, Maryland; B. F. SINE, Principal of High School, Rock Enon Springs, Virginia; and PROPOSER.

Let a and b be the bases, p the perpendiculars, and A the angle between the diagonals.



Take $BC=a+b$ and describe upon BC a segment to contain an angle$=$to A. The problem is possible when p is less than the greater segment of the diameter perpendicular to BC. Take $CE=p$ and perpendicular to BC. Draw EH parallel to BC cutting the circle in M and G. Draw BG and GC. Also draw DF parallel to BG and DH parallel to GC.

Then is $DCFG$ or $BDGH$ the required trapezoid. For $BD=GF=b$, $DC=HG=a$, $\angle DKC=\angle BLD=\angle BGC=A$, and $CE=p$. By treating the point m as we did G we get two other trapezoids answering all conditions.

This problem was solved in a similar manner by *COOPER D. SCHMITT, A. H. BELL, J. SCHEFFER, B. F. SINE, J. M. COLAW, P. S. BERG, O. W. ANTHONY, E. W. MORRELL, J. C. GREGG,* and *H. J. GAERTNER.*

PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A fly starts from a point in the circumference of a table, 3 feet in diameter, and travels uniformly along the diameter to a point in the circumference of the table directly opposite the starting point. The table moves uniformly to the right about a center axis in such manner that it makes one complete revolution while the fly passes over its diameter. Find the absolute path described by the fly and the ratio of rates of movement of the table and the fly.

I. Solution by the PROPOSER.

The curve described by the fly is the spiral of Archimedes. Its equation is $r=a\theta$. $S=\int_0^\pi \left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta, =\frac{a\pi\sqrt{1+\pi^2}}{2}+\frac{a}{2}\log(\pi+\sqrt{1+\pi^2})$.

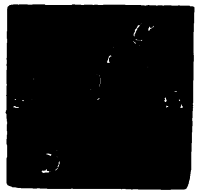
Hence, $2S$, or the absolute path described by the fly, is 63.994+ inches.

If we take the Naperian logarithm of $(\pi+\sqrt{1+\pi^2})$ the result is 69.6+ inches.

The ratio of rates$=\dfrac{2\pi r}{2r}=\pi$. The ratio of rates in *space*$=\dfrac{2\pi r}{63.994}=1.76+$.

II. Solution by G. B. M. ZERR, Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas; and Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

Let P be the position of the fly when A has moved to C, and let A move m times as fast as P. Let $OA=r$, $OP=\rho$, $\angle COA=\theta$.
Then $mPC=AC$. $\therefore$ $m(r-\rho)=r\theta$.



$\therefore$ $\rho=\dfrac{r(m-\theta)}{m}=\dfrac{r(\pi-\theta)}{\pi}$, since $m=\pi$. This is the equation to the fly's path.

$$\therefore\ S=\int_0^{2\pi}\frac{r}{\pi}\sqrt{1+(\pi-\theta)^2}\,d\theta$$

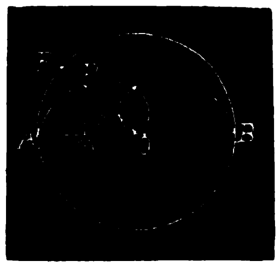
$$=r\sqrt{1+\pi^2}+\frac{r}{\pi}\log\left(\pi+\sqrt{1+\pi^2}\right).$$

$$\therefore\ S=\tfrac{3}{2}\left\{\sqrt{1+\pi^2}+\frac{1}{\pi}\log\left(\pi+\sqrt{1+\pi^2}\right)\right\}=5.835 \text{ feet}.$$

$$\frac{3\pi}{S}=\frac{1885}{1167}=\frac{13}{9}\text{ nearly}.$$

III. Solution by Prof. J. M. BANDY, A. M., Old Trinity College, North Carolina, and J. C. GREGG, Superintendent of City Schools, Brazil, Indiana.

Let (ρ, θ) denote the co-ordinates of P, and since AR and RP are in a constant ratio, ρ and θ are in the same ratio, which denote by c.



Hence, $\theta=-\rho c$ [Archimedean spiral]......(1).
By theory of curves,

$$S=\int\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}d\theta\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $\dfrac{d\rho^2}{d\theta^2}=\dfrac{1}{c^2}$, and $\rho=\dfrac{\theta^2}{c^2}$. Substituting these values in (2), $S=\dfrac{1}{c}\displaystyle\int_0^{2\pi}(1+\theta^2)^{\frac{1}{2}}\,d\theta\ldots\ldots\ldots\ldots\ldots\ldots(3).$

Integrating (3) by formula for reducing $p=1/2$,

$$S=\left[\frac{\theta(1+\theta^2)^{\frac{1}{2}}}{2c}\right]_0^{2\pi}+\frac{1}{2c}\log\left[\theta+\sqrt{1+\theta^2}\right]_0^{2\pi}\ldots\ldots\ldots\ldots(4).$$

But $c=\frac{\text{arc } AR}{RP}=\frac{\text{circumference}}{\text{diameter}}=\frac{\pi}{1}$. Substituting in (4), and reducing, $S=6.4533+$ feet.

The movement of the fly in its path is the resultant of the motion of the fly along the diameter and the motion of the table to the right about its axis. The rate of motion of the fly in its path is variable, and is measured at any instant by the measuring circle given by any particular value of ρ. So that the ratio of the motion of the table to that of the fly can be found for any particular value of ρ.

Also solved by *O. W. ANTHONY* and *E. L. SHERWOOD*. Prof. Sherwood's solution will be published under problem No. 50.

46. Proposed by H. C. WHITAKER, M. E., Sc. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

There are four points, A, B, C, and D in space. Point D remains fixed with its co-ordinates (1, 2, 2) feet. At a given time A is at (2, 3, 4) feet, is moving in a straight line at the rate of 3 feet per minute, and has passed through (5, 9, 10) feet; B is at (1, 4, 2) feet, moves in a straight line at the rate of 7 feet per minute, and will pass through (—2, 2, 8) feet; C is at the origin and moves along the axis of X in the direction of x positive at the rate of 6 feet per minute.

The motion of the points being continuous before and after the given time, required the times when the volume of the tetrahedron whose edges are the lines joining these points will be 108 cubic inches.

Solution by the PROPOSER.

The length of a base edge [from (x_1, y_1, z_1) to (x_2, y_2, z_2)] is well known to be

$$\sqrt{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}^2+\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_3y_3z_3)$ to this edge, multiplying this distance by the length of the edge just given, the area of the base is

$$\frac{1}{\lfloor 2}\sqrt{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2+\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_4y_4z_4)$ to this base, multiply this distance by the area of the base just given, the volume of the tetrahedron is found to be *

$$\frac{1}{\lfloor 3}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

[*The extension of each of these values in n dimentioned co-ordinates is obvious, as is also the sursolidity of a figure in four dimensioned space bounded by five tetrahedra; and so on.]

Substituting the given values of the co-ordinates, we have

$$\frac{1}{6}\begin{vmatrix} 2-t & 3-2t & 4-2t & 1 \\ 1-3t & 4-2t & 2+6t & 1 \\ 6t & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{vmatrix}=V.$$

This reduces to $16t^3-14t^2+2t=\pm.0625$; whence by solving, $t=-.026$, $.045$, $.125$, $.217$, $.684$, and $.705$ seconds, respectively.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

PROBLEMS.

53. Proposd by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$; and show that $x=y\log(x\,y)$.

54. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite sides of the square are semi-circles, however. Find surface, volume, and the centers of gravity of each.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PLAYFAIR'S PSEUDO-PROOF OF THE ANGLE-SUM.

BY GEORGE BRUCE HALSTED.

The living person who has most capital invested in Playfair's fallacious demonstration reproduced in the March number of THE AMERICAN MATHEMATICAL MONTHLY, pp. 77—79, is Professor George C. Edwards of the University of California, who unfortunately gives it as the basis for his treatment of parallels in §16 of his Elements of Geometry, Macmillan, 1895.

His §16 is Playfair's Proposition I "All the exterior angles equal four right angles," with Playfair's fallacious proof. Then his §17 is "THEOREM. If two straight lines make equal angles with a third straight line intersecting them, they will make equal angles with *any* straight line intersecting them," in proving which he twice cites §16. Then as Exercise 1 under §17 he has "Establish the theorem when the fourth line passes through B." But this very special case of his §17 he assumes in his §16, thus making his treatment of parallels a simple

argumentum in circulo. I wrote this to Professor Edwards and he wrote in answer what clearly seemed an explicit acknowledgment of it. But it was so unlike a paradoxer to acknowledge a fallacy, that in wonder I wrote again, "You mean to state that in your proof of the theorem §16 of your book, you do assume (without stating the assumption) your Exercise 1 under your §17. Am I right in this understanding of your letter?" And strange as it may seem he wrote March 7th, 1896, "You are practically right in your understanding of my letter of February 22nd."

I have given three different exposures of Playfair's fallacy in the fourth edition of my Bolyai pp. 65—71.

THEORY AND PRACTICE COMBINED.

BY WARREN HOLDEN, GIRARD COLLEGE, PHILADELPHIA, PENNSYLVANIA.

Common experience, applied to Mechanical and Engineering problems, has always been in harmony with the principles of Euclidean Geometry. With the overthrow of these principles we might expect chaos to come again. And if Mathematics has not yet demonstrated all of these principles, so much the worse for Mathematics. Let its Professors try again. Their failure in any particular case does not establish the opposite.

Abstract studies in Philosophy, unmodified and unillustrated by human experience, have often led to bewildering vagaries. Does not a similar fate, from corresponding causes, impend over Non-Euclidean Geometry? Theory and practice should go hand in hand.

All mathematical instruments in use, whether in the department of Mechanics, Physics or Engineering, are constructed upon the basis of Euclidean Geometry. Where are the instruments of precision which serve to illustrate and apply the principles of Non-Euclidean Geometry?

QUERIES.

1. Please give me address of publishing house that publishes the most reliable works on How to Calculate Timber on the Stump, also names of most reliable works on same.

JOHN BRIDGES.

EDITORIALS.

Prof. C. A. Waldo is now Professor of Mathematics in Purdue University, Lafayette, Indiana.

Science, March 27, contains an able article, The Essence of Number, by Dr. George Bruce Halsted.

Prof. J. A. Calderhead has been elected Professor of Mathematics in the Curry University, Pittsburg, Pennsylvania.

Dr. Byerly's Fourier's Series and Spherical Harmonics, we are informed by the Publishers, is gaining an international reputation.

Dr. E. H. Moore has been promoted to Head Professor of Mathematics in the University of Chicago. This is a merited recognition.

Professor J. J. Sylvester, formerly of the Johns Hopkins University, has just been made a Foreign Member of the Turin Royal Academy of Science.

Our subscribers will do us a kindness by sending us the names of persons who are likely to subscribe for the MONTHLY, as we would be pleased to send such persons sample copies.

A few of our former subscribers who are in arrears have asked us to discontinue the MONTHLY to their address. In no case will we discontinue to send the MONTHLY until the amount due us is paid.

Mr. W. J. C. Miller, who is editor of the Mathematical Department of the *Educational Times*, London, England, says, "THE AMERICAN MATHEMATICAL MONTHLY is one of the best magazines that I receive." Mr. Miller has edited the Mathematical Department of the *Educational Times* for over 30 years.

M. A. Gruber, of Washington, D. C., writes: You will please find enclosed a Money Order of $3.00 as my subscription to THE AMERICAN MATHEMATICAL MONTHLY for 1896. It is a magazine worthy of long life; if the additional mite is any assistance in putting it upon a paying basis, I shall always remain among your best friends.

We have on hand a few bound copies of Volumes I and II which we will sell at $2.75 each. By special arrangements with the binders we can have volumes of the MONTHLY bound for 75 cents. If any of our subscribers wish to avail themselves of this opportunity to have their volumes of the MONTHLY bound, they may send them to B. F. Finkel, Springfield, Mo.

Philadelphia Summer Meeting will hold its fourth session, July 6—31, 1896, in the buildings of the University of Pennsylvania, under the auspices of the American Society for the Extension of University Teaching. Department E —Mathematics: I. Methods of Teaching Mathematics; II. Plane and Solid Geometry; III. Algebra (Elementary Course); IV. Algebra (Advanced Course); V. Trigonometry; VI. Analytical Geometry; VII. Differential and Integral Calculus; VIII. Theory of Equations and Determinants; IX. Differential Equations; X. Theory of Functions.

The lecturers are I. J. Schwatt, Ph. D., and G. H. Hallett, M. A., of the University of Pennsylvania. On Wednesday evening, July 8, Dr. Schwatt will deliver to the students of all departments of the Summer Meeting an address on the Philosophy and Utility of the Calculus.

We are sorry to announce the death of one of our valued contributors, T. P. Stowell, of Rochester, N. Y., which occurred February 29th, 1896. Mr.

Stowell's name has been closely associated with nearly all the mathematical journals published in this country within the last fifty years. The following sketch is taken from *The Union and Advertiser*, Rochester, New York:

Thomas P. Stowell, of No. 29 Atkinson street, died Saturday at the home of the family, aged 77 years. Mr. Stowell, who had resided in the city since April 1, 1864, at the residence now occupied by the family, was born September 5, 1819, and was widely known, respected and esteemed, not only in Rochester but throughout the entire country. He graduated from the well-known Hallowell University of Virginia, and was considered one of the ablest mathematicians in the United States. He retired from business in 1895, in the enjoyment of robust health, having apparently the strength and certainly the appearance of a middle-aged man.

Mr. Stowell had been a member of St. Luke's Church during the entire period of his residence in Rochester. He leaves a wife and five children, Miss Anna Stowell, Miss M. Louise Stowell, Dr. Henry F. Stowell, and C. L. Stowell, all of this city, and Charles F. Stowell of Albany.

BOOKS AND PERIODICALS.

Syllabus of Geometry. By G. A. Wentworth, A. M., Author of a Series of Text-books in Mathematics. Pamphlet form. 50 pages. Boston and Chicago: Ginn & Co.

This pamphlet contains the enunciations of the propositions and corollaries of the author's text-book in Geometry, numbered as they are in the text-book. B. F. F.

Rational Mathematics. By Charles De Medici.

Under the above title the author is publishing a work—The New Geometry and Commensurational Arithmetic—which is divided into three sections: A, B, C. In Section A, Part I, the first principles and primary elements of Geometry are taught; Part II. First principles of Commensuration, founded on the Natural Division and Inherent Dimensions of Geometric Elements are taught; Part III. Classification of Geometric Figures and Forms. Section B, Geometry Study and Practice. The work is published by A. Lovell & Co., New York. B. F. F.

Elementary Treatise on Electricity and Magnetism Founded on Joubert's Traité Élémentaire D'Éléctricité. By G. C. Foster, F. R. S., Quain Professor of Physics in University College, London, and E. Atkinson, Ph. D., formerly Professor of Experimental Science in the Staff College. 8vo. Cloth, 552 pp. Introduction price, $1.80. New York: Longmans, Green & Co.

This treatise on Electricity and Magnetism is confined to facts, hypotheses being studiously avoided. The treatment of each subject is clear, simple, direct, and exhaustive. Whenever necessary, the higher mathematics are used in computations and the establishment of electrical laws. It is the best treatise on Electricity and Magnetism that we have yet seen and we heartily commend it to any person desiring a good work on these important subjects. B. F. F.

Elementary Algebra. By J. A. Gillett, Professor in the New York Normal College. 8vo. Half Leather Back. 412 pp. New York: Henry Holt & Co.

Among other commendable features of this book may be mentioned, (1) the prominence given to problems and the consequent introduction of the equation, (2) the attention given to negative quantities, (3) the attention given to the formal laws of Algebra,—the Commutative, the Associative and the Distributive laws, and (4) the simplicity, clearness, and logical arrangement of the matter. The book is beautifully printed and handsomely bound, and presents a most attractive appearance.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number 25 cents. The Review of Reviews Co., New York City.

The *Review of Reviews* is almost indispensable to the general reader who wishes to keep abreast of the rapidly developing international questions of the day. In the April number there is a full and able editorial discussion of the complicated African situation, which is described as "the drama of 'Europe in Africa.'" The mixed interests and motives of England, Russia, Italy and France in the Dark Continent are clearly set forth. Russia's general attitude toward the European powers is also discussed, and the editor comments briefly on America's relations with Spain, our interests in the Cuban revolution, and the present status of the Venezuelan boundary dispute. In addition to this editorial treatment (in the department entitled "The Progress of the World") the *Review* presents a remarkably complete survey of the Cuban situation by Murat Halstead, a summary of the best current thought in England on the subject of international arbitration, and a vivid account of the relief work now going on in Armenia. In short the *Review of Reviews* records a month's activities in both hemispheres.

April Monthly Magazine Number of the Outlook. Price, $1.00 per year in advance. The Outlook Company, 13 Astor Place, New York.

In the April Magazine Number of *The Outlook* there will appear an article on William H. Prescott, by Kenyon West. It will be in commemoration of the centenary of the great American historian, who was born May 4, 1796. The article will be enriched by uumerous portraits and other illustrations contirbuted from the private collection of members of the Prescott family, who have been interested in Kenyon West's tribute to Prescott. Among these are Mr. Arthur Dexter, of Boston, the nephew of the historian; Mrs. Roger Wolcott, Prescotts grand-daughter, who lives also in Boston; and Mr. Linzee Prescott of Greenwich, Conn., who is the son of Prescott's eldest son.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The April *Cosmopolitan* contains the following: A word about Golf, Golfers, and Golf-links in England and Scotland, by Price Collier; Vicissitudes of the Dead, by Eleanor Lewis; Development of the Overland Mail Service, by Thomas L. James; The Lyceum, by James B. Pond; Mrs. Cliff's Yacht, by Frank R. Stockton; The Bargain of Faust (Poem) by Alice W. Rollins; Hilda Stafford, by Beatrice Harraden. Each of these articles are beautifully illustrated.

The following periodicals have been received: Journal de Mathématiques Élémentaires, (15 Mars 1896); American Journal of Mathematics, (April, 1896); The Mathematical Gazette, (October, 1895); L'Intermédiaire des Mathématiciens, (Mars, 1896); El Progreso Matemático, (Tomo V. Ano 1895); Notes and Queries, (April, 1896); The Kansas University Quarterly, (January, 1896); Popular Astronomy, (June, 1895); The Monist, (April, 1896); Bulletin of the American Mathematical Society, (March, 1896); The Educational Times, (March, 1896).

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A SPECIAL COMPLEX OF THE SECOND DEGREE AND ITS RELATION WITH THE PENCILS OF CIRCLES.

By DR. ARNOLD EMCH, University of Kansas, Lawrence, Kansas.

1. Before entering upon the treatment of this problem I will make a few preliminary remarks which, although well known to the reader, may give a clearer conception of what follows.

If a straight line is given we can write the equations of its projections upon the co-ordinate planes in the following form :*

$$(1)\qquad \begin{aligned} L&=yZ-zY,\\ M&=zX-xZ,\\ N&=xY-yX, \end{aligned}$$

where x, y, z designate the current co-ordinates. The six constants L, M, N, X, Y, Z can be considered as the co-ordinates of the straight line and satisfy the relation

$$(2)\qquad LX+MY+NZ=0.$$

An algebraic complex of straight lines of the n^{th} degree is defined by an equation of the form

$$(3)\qquad F(L,\ M,\ N,\ X,\ Y,\ Z)=0,$$

F being a polynom of the n^{th} degree and homogeneous in L, M, N, X, Y, Z.

Through every point in spaces passes an infinite number of lines belonging to the complex ; they form a cone of the n^{th} order, and in every plane lies an infinite number of lines belonging to the complex and enveloping a curve of the

*We use the designation of M. Picard in his *Traite d'Analyse*, Vol. I, p. 312.

n^{th} class. Thus, a special kind of a complex of the n^{th} degree may be obtained by all the tangents of a surface of the n^{th} order, or the secants of a curve in space of the n^{th} order.

2. In our problem we define as a special complex P of the second degree the system of secants passing through a fixed conic. Every point in space determines a cone of the second order, whose elements belong to the complex and every plane intersects the conic in two points which represent a degenerated curve of the second class, whose tangents belong to the complex.

The conic itself we will describe in the following manner:

Through any two fixed points A and B of the xy-plane draw the two circles

$$(4) \qquad U_1=(x-a_1)^2+(y-b_1)^2-r_1{}^2=0,$$

$$(5) \qquad U_2=(x-a_2)^2+(y-b_2)^2-r_2{}^2=0,$$

and form the pencil of circles

$$(6) \qquad U_1-\lambda U_2=0$$

passing through A and B. At the center of every circle of the pencil erect a perpendicular to the xy-plane and equal to the radius of the circle above and below the xy-plane. The extremities of these perpendiculars lie in an equilateral hyperbola H whose plane passes through the central line of the pencil of circles and is perpendicular to tne xy-plane. The vertices of the hyperbola are equal distant from the xy-plane and lie in a perpendicular through the center of the circle with the sect AB as a diameter.*

To every point of the equilateral hyperbola belongs a circle of the pencil (6), which with the point determines a right cone whose elements all include angles of 45° with the xy-plane. We may ask what is the character of the system of lines R passing through the equilateral hyperbola and including angles of 45° with the xy-plane. For this purpose intersect the *cone-director* of these lines with the plane at infinity and establish the new complex Q consisting of all the lines passing through the intersection. As the intersection is a circle I, the complex is of the second degree and contains all the lines including angles of 45° with the plane xy.

Evidently the system R is the common solution of the complex P and Q and is therefore a *congruence*. The degree of this congruence is 6, since the degrees of P and Q are 2, and since the hyperbola H and the circle I have two points in common. Through each point in space pass two lines, and in each plane lie four lines belonging to the congruence. It is therefore of the second order and of the fourth class. Since the equilateral hyperbola H is symmetrical in regard to the xy-plane, it is easily seen that the complex and the congruence connected with it are symmetrical to the xy-plane, in other words they are reflected into themselves. It is known that through every generatrix of a congruence of straight lines pass two developable surfaces whose elements belong to

*The thought to represent points in space by circles in a plane originated with Prof. W. Fiedler, of Zurich, who applied it in his beautiful treatise on "*Cyclographie*," Teubner, Leipzig.

the congruence. In our congruence R the developable surfaces through a generatrix D are the right cone having its vertex in the hyperbola H and the hyperbolic cylinder passing through H. The focal surface of the congruence degenerates into the hyperbola H and the plane at infinity. If we designate the representation of a pencil of circles by Fiedler's method, and the complex and congruence of rays connected with it as *cyclographic*, we may now state the theorem :

The theory of the pencils of circles is identical with the theory of the cyclographic congruence.

3. To the first pencil of circles through A and B, or cyclographic congruence; we add another pencil of circles through the points C and D, or cyclographic congruence. It may be determined by two circles

$$(7) \qquad V_1=(x-c_1)^2+(y-d_1)^2-s_1{}^2=0,$$

$$(8) \qquad V_2=(x-c_2)^2+(y-d_2)^2-s_2{}^2=0,$$

passing through C and D and assumes the form

$$(9) \qquad V_1-\mu V_2=0.$$

The corresponding congruence is obtained as in the first pencil. Designating this congruence by S and the complex through the hyperbola G which represents the pencil (9) by T we have to solve the problem to find the common part of the congruences R and S, or as these have the circle I at infinity in common, to find the common figure of the complexes P, Q, and T. Each of the hyperbolas H and G intersect the circle I in two points and as the complexes are all of the second degree, they have a ruled surface in common whose degree according to the rules of algebra is

$$2\times2\times2\times2-2\times2-2\times2=8.$$

To a generatrix in this ruled surface of the eighth order can be found one in the same surface symmetrical to the first in regard to the xy-plane. Hence the whole surface is symmetrical to the xy-plane and as it contains two double generatrices through the circular points of the circle I, it intersects the xy-plane in a bicircular curve of the fourth order. Every generatrix of the surface intersects the xy-plane in a point of the curve and includes an angle of 45° with the xy-plane.

Through each generatrix pass four developable surfaces, two hyperbolic cylinders and two cones of the second order. These cones are tangent to each other and intersect the xy-plane in two tangent circles. As these circles always pass through A, B and C, D and as their point of tangency lies in the above curve we have the theorem :

The locus of the points of tangency of each two tangent circles of two pencils of circles is a bicircular curve of the fourth order.

Figure 1 will show the relation of these pencils in the case that each two circles are tangent.

4. We will now take another view of the problem. For fixed values of λ and μ the equations of two circles respectively belonging to the pencil (6) and (9) may be written

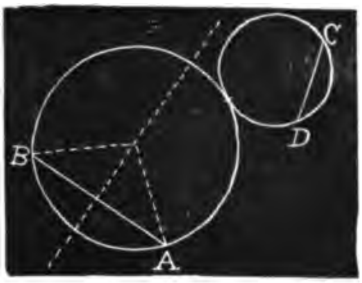


Fig. 1.

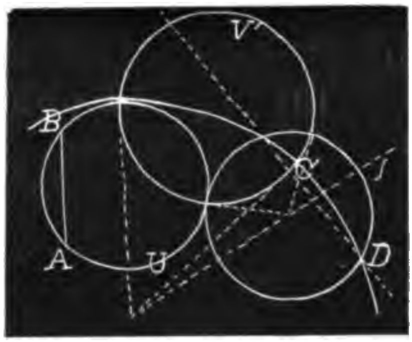


Fig. 2.

$$(10) \qquad x^2+y^2-2\frac{a_1-\lambda a_2}{1-\lambda}x-2\frac{b_1-\lambda b_2}{1-\lambda}y+\frac{M_1-\lambda M_2}{1-\lambda}=0,$$

$$(11) \qquad x^2+y^2-2\frac{c_1-\mu c_2}{1-\mu}-2\frac{d_1-\mu d_2}{1-\lambda}y+\frac{N_1-\mu N_2}{1-\mu}=0,$$

where $\qquad M_1=a_1^2+b_1^2-r_1^2, \; M_2=a_2^2+b_2^2-r_2^2,$

$$N_1=c_1^2+d_1^2-s_1^2, \; N_2=c_2^2+d_2^2-s_2^2.$$

The condition that the circle (10) is orthogonal to the circle (11) is

$$2\frac{a_1-\lambda a_2}{1-\lambda}\cdot\frac{c_1-\mu c_2}{1-\mu}+2\frac{b_1-\lambda b_2}{1-\lambda}\cdot\frac{d_1-\mu d_2}{1-\mu}-\frac{M_1-\lambda M_2}{1-\lambda}-\frac{N_1-\mu N_2}{1-\mu}=0.$$

It is now possible to determine the co-efficients of this equation such that for variable parameters the pencils (10) and (11) are projective. In this case each two corresponding circles are orthogonal.

Evidently we have to put $\mu=\lambda$, which after some reductions gives for the equation of condition

$$[2a_1c_1+2b_1d_1-M_1-N_1]-\lambda[2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2]$$
$$+\lambda^2[2a_2c_2+2b_2d_2-M_2-N_2]=0. \qquad (12)$$

This indicates that in the first place the circles (4) and (7), and (5) and (8) must be orthogonal. Secondly, for every value of λ there must be

$$2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2=0.$$

This equation is satisfied if the circles (5) and (7), and (4) and (8) are orthogonal. In this case the pencils (6) and (9) are said to be conjugate pencils of circles. Every circle of the one pencil is orthogonal to every other circle of the other. But in equation (12) we do not desire to change the points A, B, C, D. Thus, the equation can be satisfied by changing the radii of the circles (4), (5), (7), (8), which gives for the solutions of M_1, M_2, N_1, N_2 three equations with four unknown quantities. We can however fix one of the circles without altering the final result. This implies the theorem :

Two pencils of circles can be made projective in one and only one way such that corresponding circles in the projectivity are orthogonal.

The product of these projective pencils is a bicircular curve of the fourth order, as it is well known. In figure 2, we consider the two pencils of circles through A and B, and C' and D', where C' and D' are assumed to be imaginary and on the line l and in these pencils two orthogonal circles U and V' intersecting each other in two points J and J'. In these points draw tangent circles to U having their centers on l. These circles are orthogonal to V' and intersect each other in two fixed points C and D, *i. e.*, they belong to the conjugate pencil of circles of the pencil through C' and D'. Whence the general theorem:

The locus of the points of tangency of each two tangent-circles of two pencils of circles is a bicircular curve of the fourth order. The same curve is also produced by one of the pencils and the projective conjugate pencil of the other pencil.

Under the given conditions the equation of the curve may be written $U_1V_2'-U_2V_1'=0$.

It is easily seen that this curve passes through the four points A, B, C, D and as stated in the theorem contains the circular points at infinity as double-points.

5. Without entering into further details on the nature of this curve it may be mentioned that there exists an interesting connection between this curve and the circular curves of the third order if these are considered as locii of points from which two sects AB and CD appear under the same angle. An analogon exists in space, the discussion of which however goes over the limits of this article. A paper on this subject by the author was read in the January session of the Kansas Academy of Science and will appear in the next volume of the transactions of this Academy.

PROBLEMS.

1. Given n straight lines in a plane. Another straight line in this plane revolves about a fixed point and in every position intersects the n lines

in n points. These points determine n "sects" measured from the fixed point and their algebraic sum represents a point on the revolving line. What is the curve which this point describes?

2. Find a geometrical construction for the following problem: Given the distances AO, BO, CO of the points A, B, C of an equilateral triangle from a fixed point O. Construct the equilateral triangle, or triangles satisfying these conditions.

3. What is the locus of the points from which any two sects in space AB and CD (not in the same plane) appear under the same constant angle?

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from April Number.]

PROPOSITION XXIV. *The same hypothesis remaining: I say the four angles together* (*Fig.* 27.) *of the quadrilateral* $KDHK$ *nearer the base* AB *are less* (*in hypothesis of acute angle*) *than the four angles together of the quadrilateral* $KHLK$ *more remote from the same base; and indeed this is so, whether those two* AX, BX *somewhere at a finite distance meet toward the parts of the point* X*; or never meet one another; but toward those parts either ever more mutually approach each other, or somewhere receive a common perpendicular, after which of course* (*in accordance with Cor. II. of the preceding proposition*) *toward the same parts they begin mutually to separate.*

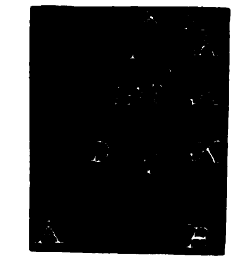


Fig. 27.

PROOF. Here however we suppose the portions KK assumed to be mutually equal. Since therefore (from the preceding) the side DK is greater than the side HK, and similarly HK greater than the side LK, the portion MK in HK is assumed equal to LK, and in DK the portion NK equal to HK; and MN, MK, LK are joined, truly the intermediate point K with the point L, and the point K near to the point B with the point M.

Now I proceed thus.

Since indeed the sides of the triangle KKL (I make beginning always from the point K nearer the point B) are equal to the sides of the triangle KKM, and the included angles equal, as being right, equal also will be (from Eu. I. 4) the bases LK, MK, and likewise equal the angles which correspond mutually,

at these bases, indeed the angle KLK to the angle KMK, and the angle LKK to the angle MKK. Therefore equal also are the remainders NKM and HKL. Wherefore, since the sides NK, KM of the triangle NKM are equal in the same way to the sides HK, KL of the triangle HKL, equal also will be (from the same Eu. I. 4) the bases NM, HL, the angles KNM, KHL, and finally the angles KMN, KLH. But in the preceding triangles are already proved equal the angles KLK, KMK. Therefore the whole angle NMK is equal to the whole angle HLK.

Wherefore, since all angles at the points K are right, it follows manifestly all four angles together of the quadrilateral $KNMK$ are equal to all four angles together of the quadrilateral $KHLK$.

But since the two angles together at the points N and M in the quadrilateral $KNMK$ are greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI) at the points D and H in the quadrilateral $NDHM$, or the quadrilateral $KDHK$, the consequence thence is, that (the common right angles at the points K being added) the four angles together of the quadrilateral $KNMK$, or the quadrilateral $KHLK$ are greater (in hypothesis of acute angle) than the four angles together of the quadrilateral $KDHK$.

Quod erat demonstrandum.

COROLLARY.

But it ought here opportunely to be observed, nothing will fail in the argument made, although the angle at the point L is assumed right, together with hypothesis of acute angle. For still that common perpendicular LK would be less (from Cor. I. after II of this) than the other perpendicular HK, from which therefore still a portion MK could be assumed equal to the aforesaid LK.

Which standing, it follows that no hindrance can intervene.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from April Number.]

PRIMITIVE GROUPS OF TWO, THREE, AND FOUR, LETTERS.

Since all of these must contain substitutions of the form a_1a_2 or of the form $a_1a_2a_3$ they must all contain the symmetric group. The following is therefore a complete list:

Degree.	Order.	Group.
2	2	(ab)
3	3	(abc)
	6	(abc)all
4	12	$(abcd)$pos
	24	$(abcd)$all.

PRIMITIVE GROUPS OF FIVE LETTERS.

All the transitive groups of this degree must be primitive and there must be one regular group, viz:

(1) $(abcde)$.

The lowest order of any other possible primitive group is 10. Such a primitive group would contain five subgroups of the form $(ab.cd)$ and therefore four substitutions of degree 5. All the substitutions of degree 5 whose powers do not contain a substitution of the type a_1a_2 are of the type $a_1a_2a_3a_4a_5$. Hence all the substitutions of degree 5 of a primitive group which is not the symmetric group must be of the given type.

If a primitive group of order 10 exists we may therefore assume that it contains

$$(abcde)$$

and some substitutions of the form $ab.cd$. These substitutions are all equal to

$$(abcde)S$$

where S is any one among them. They therefore transform the substitutions of $(abcde)$ into the same power. This cannot be the first power for a substitution consisting of a single cycle can be transformed into the first power only by its own powers. If we represent this power by α and observe that the product of two of these supstitutions is equal to some substitution in $(abcde)$ we have*

$$\alpha^2 \equiv 1 \pmod 5), \ (1 < \alpha < 5).$$

Since this has only one solution it follows that there is only one group of order 10. We may find the substitutions by writing the fourth power of $abcde$ under $abcde$, thus,

*If s_1 and s_2 transform s into s^a we have

$$s_1^{-1}ss_1 = s^a$$

$$s_2^{-1}s_1^{-1}ss_1s_2 = s_2^{-1}s^a s_2 = s_2^{-1}ss_2 . s_2^{-1}ss_2 \ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^a . s^a . s^a \ldots\ldots\ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^{a^2}$$

abcde	*abcde*	*abcde*	*abcde*	*abcde*
aedcb	*baedc*	*cbaed*	*dcbae*	*edcba.*

The required substitutions are

$$be.cd \qquad ab.ce \qquad ac.de \qquad ad.bc \qquad ae.bd$$

[We might evidently have obtained all of these by multiplying one into $(abcde)$]. Hence the group of order 10 is

(2) $$(abcde)(ab.ce)=(abcde)_{10}.$$

All the substitutions that transform $(abcde)$ into itself form a group. There are five substitutions that transform the substitutions of $(abcde)$ into their first power, therefore there must be five that transform them into each of their other powers. We thus obtain a group of order 20 which is generated by $abcde$ and some substitution $bced$ which transforms this into its second power. We have therefore

(3) $$(abcde)(bced)=(abcde)_{20}.$$

There cannot be more than one of this order because each would have to contain five conjugate subgroups of one of the two types

$$(abcd)_4, \ (abcd)$$

and therefore only one subgroup (necessarily self conjugate) of the type

$$(abcde).$$

This may be supposed to be the same in all of the groups; but there is only one set of twenty substitutions that transform this into itself. The groups are therefore identical.

For all the other possible orders the subgroups of degree 4 would contain either a substitution of the type ab or one of the type abc. Hence all the other primitive groups are the alternating and the symmetric group. The following is a complete list of the primitive groups of degree five.

Order.	Group.
5	$(abcde)$
10	$(abcde)_{10}$
20	$(abcde)_{20}$
60	$(abcde)$pos
120	$(abcde)$all

These groups could also have been found in the following manner, without employing the groups of a lower degree. We know that there is one group of each of the three classes—regular, alternating and symmetric. We know also that the order of each of the other primitive groups exceeds five and that they do not contain any substitutions of either of the two types

$$ab \qquad abc$$

Hence they can contain only substitutions of the fourth and fifth degrees together with unity.

Since the average number of letters in all the substitutions of these groups must be four each group can contain only four substitutions of the fifth degree. The only type of substitutions of the fifth degree which can be used is

$$abcde.$$

All these primitive groups may therefore be supposed to contain

$$(abcde)$$

as a self-conjugate subgroup and to be subgroups of the group of order 20 which contains all the substitutions that transform $(abcde)$ into itself.

Any negative substitution of this group together with $(abcde)$ generates the entire group, the only subgroup besides the group itself and $(abcde)$ must therefore consist of the positive substitutions of the group. Hence there are only two primitive groups of degree five in addition to the regular, alternating, and symmetric groups. The generating substitutions of these groups are evident.

Primitive Groups of Six Letters.

There is no regular group. If there were a group of order 30 it would contain 24 substitutions of the type $abcde$ and five substitutions of degree six. These five substitutions would generate a regular group; for only one of them could replace a given letter by any required letter since there are four of the form $abcde$ which perform this operation, and therefore the product of any two must be of degree six or it must be unity.

[To be Continued.]

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, Instructor of Mathematics, Waco High School, Waco, Texas.

This discussion is restricted to the special cases of simultaneous quadratic equations of n variables which always admit of solution. It is assumed that solutions are always possible:

(1) When there is one equation of the second degree and one variable.

(2) When all equations except one are of the first degree.

Let q, q_1 .... q_n=terms of the second degree.

Let p, p_1 .... p_n=terms of the first degree.

Let k, k_1 .... k_n=absolute terms.

Let l, l_1 .... l_n=absolute terms.

Let m=a constant factor.

Let x, x_1 .... x_n=the variables.

Let v_1, v_2 .... v_n=the variables when the equations are transformed.

CASE 1. When one equation is general, and the rest are of the first degree, or reducible to the first degree; *i. e.* when they assume any of the following forms:

(1) $(p+k)^n=0.$

(2) $(p+k)(p_1+k_1)\ldots\ldots(p_n+k_n)=0.$

(3) $(p+k)^n+m(p_1+k_1)^n=0.$

(4) $(p+k)^{2n}+m(p+k)^n+l=0.$

In the next four cases one or more of the equations may assume the above forms instead of the forms of these cases.

CASE 2. When each equation can be resolved into two factors of the first degree and an absolute term and when one of these factors is common to all equations.

$$\left\{\begin{array}{l}(p+k)(p_1+k_1)+l_1=0.\\(p+k)(p_2+k_2)+l_2=0.\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\(p+k)(p_n+k_n)+l_n=0.\end{array}\right\}$$

Eliminate the common factor. There are now $n-1$ equations of the first degree.

CASE 3. When each equation can be resolved into two factors of the first degree and an absolute term and when each factor occurs in two equations.

As in the previous case, $n-1$ equations of the first degree can be obtained.

$$\left\{\begin{array}{l} (p_1+k_1)(p_2+k_2)+l_1=0. \\ (p_2+k_2)(p_3+k_3)+l_2=0. \\ \dots\dots\dots\dots\dots\dots \\ (p_n+k_n)(p_1+k_1)+l_n=0. \end{array}\right\}$$

CASE 4. When like terms save the terms in which one variable occurs, are equal, or can be made equal in all equations. The terms can be made equal when the co-efficients of like terms are proportional in all equations, or when they are all similar and but one occurs in each equation. By eliminating these equal terms we can obtain $n-1$ equations in which the same variable will occur in each term. By dividing by this variable, we can reduce each of these equations to the first degree.

CASE 5. When like terms of the second degree are equal, or can be made equal, in all equations. Like terms can be made equal when they meet with the requirements indicated in Case 4. Eliminate the terms of the second degree. There are now $n-1$ equations of the first degree.

CASE 6. When the equations are homogeneous and like terms save those in which one variable occurs and the absolute term, are equal, or can be made equal in all equations. The requirements for making like terms equal are given in Case 4.

$$\left\{\begin{array}{l} p_1x+q+k_1=0. \\ p_2x+q+k_2=0. \\ \dots\dots\dots\dots \\ p_nx+q+k_n=0. \end{array}\right\}$$

Let $x_1=v_1x$, $x_2=v_2x \dots\dots x_n=v_nx$. Eliminate x^2. We now have $n-1$ equations with $n-1$ variables which meet with the requirements of Case 5.

CASE 7. When two equations are homogeneous and the rest are of the first degree, or reducible to the first degree, with no absolute term. Eliminate all except two variables from the two homogeneous equations by means of the equations of the first degree. These two equations will then be homogeneous and will fall under Case 6.

CASE 8. When the terms containing one variable are equal, or can be made equal, in all equations and the remaining terms meet with the requirements of Case 6. Eliminate the terms containing this variable. We now have $n-1$ equations and $n-1$ variables which fall under Case 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor.*)

II. Solution by Henry Heaton, M. S., Atlantic, Iowa.

Since 6 oxen=10 colts, 1 ox=$1\frac{2}{3}$ colts, and 6 oxen and 11 colts=21 colts =589 sheep. $\therefore$ 1 colt=$28\frac{1}{21}$ sheep and 1 ox=$1\frac{2}{3}\times28\frac{1}{21}$ sheep=$46\frac{47}{63}$ sheep.

10 oxen and 6 colts=$22\frac{2}{3}$ colts, eat 8 acres of grass in the same time that $\frac{1}{8}$ of $22\frac{2}{3}$ colts or $2\frac{5}{6}$ colts eat 1 acre, and $3\frac{1}{3}$ colts eat an acre in the same time that 10 colts eat 3 acres. Hence $3\frac{1}{3}$ colts eat an acre in $\frac{18}{25}$ the time that $2\frac{5}{6}$ colts eat it. In $\frac{18}{25}$ the time $3\frac{1}{3}$ colts eat as much grass as $\frac{18}{25}$ of $3\frac{1}{3}$ colts or $2\frac{2}{5}$ colts would eat it in the full time. The difference between $2\frac{2}{5}$ colts and $3\frac{1}{3}$ colts is $\frac{13}{30}$ of a colt. The difference in the grass eaten by them is $\frac{7}{25}$ of the growth. Hence $\frac{13}{30}$ of a colt eats $\frac{7}{25}$ of the growth. Hence to eat all the growth will require $\frac{25}{7}$ of $\frac{13}{30}$ of a colt or $\frac{65}{42}$ of a colt=$\frac{65}{42}$ of $28\frac{1}{21}$ sheep=$43\frac{359}{882}$ sheep. To eat the growth on 9 acres will require 9 times $43\frac{359}{882}$ sheep=$390\frac{65}{98}$ sheep. $600-390\frac{65}{98}=209\frac{33}{98}$. $660-390\frac{65}{98}=269\frac{33}{98}$. Hence it will require $209\frac{33}{98}$ sheep $2\frac{6}{7}$ weeks longer to eat the original grass on 9 acres than it will $269\frac{33}{98}$ sheep to eat the same. Hence $209\frac{33}{98}$ sheep eat in the $2\frac{6}{7}$ weeks what the 60 other sheep eat in the first part of the time. Hence this time is $209\frac{33}{98}\times2\frac{6}{7}$ weeks$\div60=9\frac{1993}{2058}$ weeks. Hence it will take $269\frac{33}{98}$ sheep $9\frac{1993}{2058}$ weeks to eat the original grass on 9 acres. To eat 1 acre will require them $1\frac{1993}{18522}$ weeks.

An ox, a colt, and a sheep=$75\frac{50}{63}$ sheep.

If $75\frac{50}{63}$ sheep were eating on one acre, $43\frac{359}{882}$ sheep would eat the growth leaving $32\frac{341}{882}$ sheep to eat the original grass. If it require $269\frac{33}{98}$ sheep $1\frac{1993}{18522}$ weeks to do this, it will require $32\frac{341}{882}$ sheep $(269\frac{33}{98}\div32\frac{341}{882})\times1\frac{1993}{18522}$ weeks= $9\frac{3500813}{17664596}$ weeks.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for \$25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic.*]

I. **Solution by H. C. WHITAKER, A. M., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

If we denote taking one person one mile by a person-mile, then the total person-miles was 514 and the cost of each of them was 4.8638 cents ; the cost of taking A and B 144 miles was $7 each ; the cost of taking C 124 miles was $6.03 ; the cost of taking D 72 miles was $3.50, and the cost of taking E 30 miles was $1.46.

II. **Solution by F. M. McGAW, Bordentown, New Jersey ; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas, and H. C. WILKES, Skull Run, West Virginia.**

Five men ride 30 miles ; four, 42 miles ; three, 52 miles ; and two, 20 miles.

$\therefore$ E pays for $\frac{1}{5}$ of $30=6$ miles.

D pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42=16\frac{1}{2}$ miles.

C pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52=33\frac{5}{6}$ miles.

B pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52+\frac{1}{2}$ of $20=43\frac{5}{6}$ miles.

A pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52+\frac{1}{2}$ of $20=43\frac{5}{6}$ miles.

$144:43\frac{5}{6}=\$25:\$7.609\frac{103}{108}$, share of each A and B.

$144:33\frac{5}{6}=\$25:\$5.873\frac{91}{108}$, share of C.

$144:16\frac{1}{2}=\$25:\$2.864\frac{7}{12}$, share of D.

$144:6=\$25:\$1.041\frac{2}{3}$, share of E.

III. **Solution by A. P. REED, A. M., Clarence, Missouri, and J. C. CORBIN, Pine Bluff, Arkansas.**

144 miles = distance A rides, 144 miles = distance B rides, 124 miles = distance C rides, 72 miles = distance D rides, and 30 miles = distance E rides.

They should each pay in proportion to the distance each rides. Hence

$\frac{144}{514}$ of $\$25=\$7.00\frac{100}{257}$ = amount A should pay.

$\frac{144}{514}$ of $\$25=\$7.00\frac{100}{257}$ = amount B should pay.

$\frac{124}{514}$ of $\$25=\$6.03\frac{29}{257}$ = amount C should pay.

$\frac{72}{514}$ of $\$25=\$3.50\frac{50}{257}$ = amount D should pay.

$\frac{30}{514}$ of $\$25=\$1.45\frac{235}{257}$ = amount E should pay.

[NOTE. Greenleaf gives the answers as obtained in the second solution. But we think it is best to solve the problem on the principle that each pay in proportion to the distance he rides. This principle prevails in practice at the present time and is just in its application. EDITOR.]

PROBLEMS.

60. **Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.**

A pipe 1 foot long and $\frac{27}{37}$ inch in diameter has a half-inch orifice and weighs $1\frac{3}{4}$ pounds. What is the diameter of a pipe of the same length and orifice, but weighing 41 ounces ?

61. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

Insured my store for a/bth$=\frac{3}{4}$th part of its value, at $r=1\frac{1}{4}$ per cent.; but soon afterward the store was burned down, and my loss over the insurance was $\$L=\$4150$. What was the value of my store ?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.

Sum the series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc., to n terms; also what is the n^{th} term?

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee, and Prof. P. S. BERG, Larimore, North Dakota.

The series is evidently made up as follows from the different rows in Pascal's Triangle, beginning three farther to the right every time; thus,

a.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	......
b.				1	2	3	4	5	6	7	8	9	10	11	12	13	......
c.							1	3	6	10	15	21	28	36	45	55	
d.										1	4	10	20	35	56	84	
e.													1	5	15	35	
f.																1	
	1,	1,	1,	2,	3,	4	6,	9,	13,	19,	28,	41,	60,	88,	129,	189,	etc.

The n^{th} term of (a) is 1; the $(n-3)^{th}$ term of (b) is $n-3$; the $(n-6)^{th}$ term of (c) is $\dfrac{(n-5)(n-6)}{\underline{|2}}$; the $(n-9)^{th}$ term of (d) is $\dfrac{(n-7)(n-8)(n-9)}{\underline{|3}}$; and the $(n-12)^{th}$ term of (e) is $\dfrac{(n-9)(n-10)(n-11)(n-12)}{\underline{|4}}$; and so on. Hence the n^{th} term of the original series is composed of the sum of the above different terms; *i. e.*

$$1+(n-3)+\frac{(n-5)(n-6)}{\underline{|2}}+\frac{(n-7)(n-8)(n-9)}{\underline{|3}}+\frac{(n-9)(-10)(n-11)(n-12)}{\underline{|4}}$$

$+ \ldots\ldots$ Also, the sum of n terms of (a) is n; of $(n-3)$ terms of (b) is $\dfrac{(n-3)(n-2)}{\underline{|2}}$; of $(n-6)$ terms of (c) is $\dfrac{(n-6)(n-5)(n-4)}{\underline{|3}}$; and the sum of $(n-9)$ terms of (d) is $\dfrac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}\ldots\ldots$ and hence $S=n+$

$$\frac{(n-3)(n-2)}{\underline{|2}}+\frac{(n-6)(n-5)(n-4)}{\underline{|3}}+\frac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}+\ldots\ldots$$

Also solved by *B. F. YANNEY* and *G. B. M. ZERR.*

59. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in Michigan State Normal School, Ypsilanti, Michigan.

Prove that the product of the n n^{th} roots of 1 is $+1$ or -1 according as n is odd or even. Prove and generalize, for the n n^{th} roots of m.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. $(1)^{\frac{1}{n}}=\cos\dfrac{2m\pi}{n}+\sqrt{-1}\sin\dfrac{2m\pi}{n}$.

The several roots are: $\varepsilon^{0\sqrt{-1}}$, $\varepsilon^{(2\pi/n)\sqrt{-1}}$, $\varepsilon^{(4\pi/n)\sqrt{-1}}$, ... $\varepsilon^{[2(n-1)\pi/n]\sqrt{-1}}$.

$\therefore$ Product$=\varepsilon^{(n-1)\pi\sqrt{-1}}=\cos(n-1)\pi+\sqrt{-1}\sin(n-1)\pi=\pm1$.

If n is even product is negative; if n is odd product is positive.

II. Let $m=x+y\sqrt{-1}$.

Then $(x+y\sqrt{-1})^{\frac{1}{n}}=(\sqrt{x^2+y^2})^{\frac{1}{n}}\left[\cos\left(\dfrac{2m\pi+\theta}{n}\right)+\sqrt{-1}\sin\left(\dfrac{2m\pi+\theta}{n}\right)\right]$,

where $\theta=\tan^{-1}\dfrac{x}{y}$.

$$R_1=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{(\theta/n)\sqrt{-1}},$$

$$R_2=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(2\pi+\theta)/n]\sqrt{-1}},$$

$$R_3=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(4\pi+\theta)/n]\sqrt{-1}},$$

....

$$R_n=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{\{[2(n-1)\pi+\theta]/n\}\sqrt{-1}}.$$

$$P=\sqrt{x^2+y^2}\,\varepsilon^{[(n-1)\pi+\theta]\sqrt{-1}}=\left[\cos\left((n-1)\pi+\theta\right)+\sqrt{-1}\sin[(n-1)\pi+\theta]\right]$$

$$\sqrt{x^2+y^2}=\pm\sqrt{x^2+y^2}\left[\cos\theta+\sqrt{-1}\sin\theta\right].$$

II. Solution by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

I. $x^n-1=0$ is the equation from which are derived the n n^{th} roots of 1. Now, since 1 in the equation is negative there is one positive real root and $(n-1)$, if any, imaginary roots, if n is odd; and there is one positive real root, one negative real root, and $(n-2)$, if any, imaginary roots, if n is even. $\therefore$ Since the

imaginary roots occur in conjugate pairs, and since the product of any two conjugate imaginaries is a positive real number, the sign of the product of the n n^{th} roots of 1 when n is odd, is $+$; and when n is even, $-$.

Furthermore, since the successive powers of the first imaginary root of 1, from the 1st to the n^{th}, give us all the n^{th} roots of 1, therefore, if we denote the first imaginary root by ω, we shall have as the product of the n n^{th} roots, $\omega.\omega^2.\omega^3\ldots\ldots\omega^n=\omega^{n\frac{n+1}{2}}$. But $\omega^n=1$. $\therefore$ $\omega^{n\frac{n+1}{2}}=+1$ when n is odd; and ±1, when n is even. But of these last two signs, $-$ must be chosen, for reasons assigned in the preceding paragraph.

II. That the theorem is true in general for the n n^{th} roots of m, is made evident when we remember that the n n^{th} roots of any number may be found by multiplying any one of the n^{th} roots of such number by the different n^{th} roots of 1. For then, we would have $m^{\frac{1}{n}}\times\omega.m^{\frac{1}{n}}\times\omega^2\ldots\ldots m^{\frac{1}{n}}.\omega^n=m\omega^{n\frac{n+1}{2}}=+m$ or $-m$, according as n is odd or even, as shown above.

Also solved by *COOPER D. SCHMITT.*

ERRATA. In numerator of the expression, in next to last line, on page 115 of last issue, for "Ra" read $\frac{Ra}{r}$; on page 117, line 4, for "$s(s-2a_2)$" read $s(s-2a_3)$; and in "Errata," for "last issue" read February issue. Also problems numbered 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, should be Nos. 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, respectively.

PROBLEMS.

68. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta+$, etc.

[*Chrystal's Algebra.*]

69. Proposed by Prof. C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots+(\pm1)^nx$, where A, B, C, ...are the binomial coefficients of the $(n+1)^{th}$ order.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

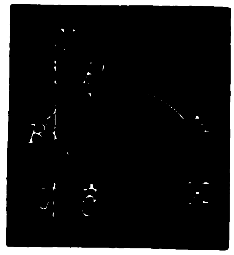
SOLUTIONS OF PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

If the *center* of a rolling ellipse *move in a horizontal line*, determine the surface on which the ellipse rolls.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let BPA be a quadrant of the ellipse semi-axes AC, and BC, O the position of the center when BC coincides with OY, and $\angle BCP=\theta$. Then



$$PC=y=\frac{ab}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{b}{\sqrt{1-e^2\sin^2\theta}}.$$

$\therefore$ The ellipse rolls on the inner surface of the cylinder

$$y^2+z^2=\frac{b^2}{1-e^2\sin^2\theta}.$$

When $e=0$, this becomes $y^2+z^2=b^2$.

To find the abscissa of the point of contact, we have, since arc PB=arc PG,

$$ds=\sqrt{r^2d\theta^2+dr^2}=\sqrt{y^2d\theta^2+dy^2} \text{ since } PC=r=y;$$

$$\text{also } ds=\sqrt{dx^2+dy^2}.$$

$$\therefore \sqrt{dx^2+dy^2}=\sqrt{y^2d\theta^2+dy^2}.$$

$$\therefore dx=yd\theta, \text{ or } x=\int yd\theta=\int\frac{bd\theta}{\sqrt{1-e^2\sin^2\theta}}=b\mathbf{F}(e,\ \theta).$$

When $e=0$, $x=b\theta$.

[No other solution of this problem was received. EDITOR.]

53. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α

and on walking to B, a distance a, directly down the hill towards the foot of the pole the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let DE be the length painted white; then a circle will pass through A, B, D, E. Let $\angle EAD = \angle EBD = \alpha$, $AB = a$, $BC = b$, $\angle DAB = \angle DEB = \theta$, $\angle ABE = \angle ADE = \varphi$, $DC = y$, and $DE = x$.

Then $(x+y)y = (a+b)b$ .... (1).

$AE : a = \sin\varphi : \sin(\alpha+\theta+\varphi)$, $x : AE = \sin\alpha : \sin\varphi$.

$$\therefore x = \frac{a\sin\alpha}{\sin(\alpha+\theta+\varphi)} \quad \text{......................(2)},$$

$b : x+y = \sin\theta : \sin(\alpha+\varphi)$ .... (3),

$(x+y) : a+b = \sin(\alpha+\theta) : \sin(\alpha+\varphi)$ .... (4).

Eliminating θ between (3) and (4),

$$\left\{\frac{(x+y)^4}{(a+b)^2} - \frac{2b(x+y)^2\cos\alpha}{a+b} + b^2\right\}\sin^2(\alpha+\varphi) = (x+y)^2\sin^2\alpha \quad \text{........(5)}.$$

Eliminating θ between (2) and (3),

$$[\{b^2x^2 - x^2(x+y)^2\}^2 + 4a^2b^2x^2(x+y)^2\sin^2\alpha]\sin^4(\alpha+\varphi)$$
$$-2a^2\sin^2\alpha(x+y)^2\{b^2x^2 + x^2(x+y)^2\}$$
$$\sin^2(\alpha+\varphi) + a^4(x+y)^4\sin^4\alpha = 0 \quad \text{................(6)}.$$

Eliminating $\sin(\alpha+\varphi)$ between (5) and (6) we get an equation in x and y which with (1) gives us the value of x.

Solved with result in terms of *EC* by *A. H. HOLMES*, and *FREDERICK R. HONEY*.

PROBLEMS.

58. Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apollonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

59. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2acxz+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

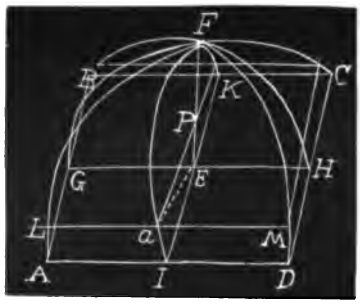
SOLUTIONS OF PROBLEMS.

47. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

I. Solution by C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland; A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

Let $ABCD$ represent the base square, side$=2a$, and KEI and GFH the two equal semi-circles, radius$=a$. Let $LMNO$ be another square parallel to the base square, and at the distance $PE=x$ from it. The area of $LMNO$ is$=4(a^2-x^2)$,



$$\therefore\ \text{Vol.}=4\int_0^a(a^2-x^2)dx=\tfrac{8}{3}a^3.$$

Denoting $\angle PEQ$ by θ, we have for the surface

$\int_0^{\frac{1}{2}\pi}8a\cos\theta d(a\theta)=8a^2$. Or for the volume, $dV=4a^2\cos^2\theta dx$, where x is the vertical distance, $x=a\sin\theta$; $dx=a\cos\theta d\theta$.

$$\therefore\ V=4a^3\int_0^{\frac{1}{2}\pi}\cos^3\theta d\theta,\ =a^3\int_0^{\frac{1}{2}\pi}(\cos3\theta+3\cos\theta)d\theta=\frac{8a^3}{3}.$$

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

The convex surface of the vault is equivalent to the surface of a right cir-

cular cylinder intercepted by another right circular cylinder, their axes intersecting at right angles, the two cylinders being equal, and the diameter of each equal to that of the vertical sections of the vault.

$\therefore$ Letting the radius$=a$, $S=8a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dxdy}{\sqrt{a^2-x^2}}=8a^2$, the equations of the cylinders being $x^2+z^2=a^2$, and $x^2+y^2=a^2$.

The volume is equivalent to that of four wedges cut from the cylinder, $x^2+y^2=a^2$, by the planes, $z=0$, and $z=x$.

$$\therefore\ V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^x dxdydz=\frac{8a^3}{3}.$$

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR.*

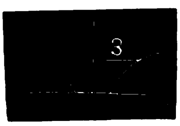
48. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

I have a circular section basin 12 inches in perpendicular height; the diameters are as follows: At base, 2 inches; one inch perpendicular height, 6 inches; two inches perpendicular height, 18 inches; three inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After a rain the water in the basin is six inches deep, what was the rainfall?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics, Mississippi Normal College, Houston, Mississippi; C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts, and the PROPOSER.

The basin is generated by revolving the curve $x=3^y$ about the axis of y.

$$\therefore\ \text{Volume of water }=\pi\int_0^6 x^2dy=\pi\int_0^6 3^{2y}dy.$$



$$\therefore\ V=\pi\frac{3^{12}-1}{2\log 3}=\frac{531440\pi}{2\log 3}.$$

Let x=depth of rain-fall, then since radius of top of basin$=3^{12}$, $V=\pi 3^{24}x$.

$$\therefore\ x=\frac{565720}{282429536481\log 3}=.00000086 \text{ inches}.$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Call x the length of any radius, and y the vertical distance, y being 1 at the bottom of the basin. Then the equation of side of basin is $x=3^{y-1}$,

$$dV=\pi x^2dy,\ V=\pi\int_1^y 3^{2y-2}dy=\frac{\pi[3^{12}-1]}{3\log_e 3}.$$

The radius of upper base$=3^{12}$. Call R the rainfall, then

$$\pi 3.^{24}R=\frac{\pi[3^{12}-1]}{2\log_e 3}.\quad R=\frac{3^{12}-1}{2.3.^{24}\log_e 3}.$$

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *B. F. YANNEY*.

ERRATA. In last issue, page 120, line 4 from bottom, for "$\rho=\frac{\theta^2}{c^2}$," read, $\rho^2=\frac{\theta^2}{c^2}$.

PROBLEMS.

55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of ground b feet in diameter. If the horse is outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? $b>a$ in each case.

56. Proposed by Prof. B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

31. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A perfectly elastic, but perfectly rough mass M, and radius R, rotating in a vertical plane with an angular velocity ω, is let fall from a height, a, upon a perfectly elastic but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let V be the vertical velocity of the center just before impact; u, v, the horizontal and vertical velocities of the center just after the first impact; ω, the

angular velocity after first impact; u' the velocity of the center just before the second impact; u_1, ω_2 the values of u, ω_1 just after the second impact, k the radius of gyration.

The equations of motion for first impact are

$$(v+V)(k^2+R^2)=2V(k^2+R^2)\dots\dots(1).$$

$$u(k^2+R^2)=\omega Rk^2\dots\dots(2).$$

The geometrical condition for no sliding is

$$u-R\omega_1=0\dots\dots(3),$$

but $V=\sqrt{2ag}$, $k^2=\frac{2}{5}R^2$.

$\therefore v=\sqrt{2ag}, u=\frac{2}{7}R\omega$, $\omega_1=\frac{2}{7}\omega$, $u'=\sqrt{v^2+u^2}=\frac{1}{7}\sqrt{4\omega^2R^2+98ag}$.

If β be the angle the center of the sphere makes with the plane just after impact we easily get

$$\cos\beta=\frac{u}{\sqrt{u^2+v^2}}=\frac{u}{u'}=\frac{2R\omega}{\sqrt{4\omega^2R^2+98ag}}.$$

Thus the motion is determined after striking the plane. Let F be the impulse arising from friction, then the equations of motion for second impact are,

$$Mu_1=Mu'\cos\beta+F\dots\dots(4),$$

$$\tfrac{2}{5}MR^2\omega_2=\tfrac{2}{5}MR^2\omega_1-RF\dots\dots(5),$$

and the geometrical condition $u_1-R\omega_2=0\dots\dots(6)$.

$\therefore F/M=-\frac{2}{7}(u'\cos\beta-R\omega_1)$, $u_1=R\omega_2=\frac{5}{7}u'\cos\beta+\frac{2}{7}R\omega_1$,

but $u'\cos\beta=R\omega_1$, $\therefore F/M=0$, and no impulsive friction is called into play after the first impact. Hence the center of the sphere describes the same parabola after each impact and the ultimate motion is the same as that after striking the plane.

III. **Solution by the PROPOSER.**

Each motion of the sphere may be considered, in its reactionary effect, separately. The motion of translation will cause the sphere to rebound after each impact to its original altitude. The time taken to attain the altitude a will be $t=\sqrt{\frac{2a}{g}}$.

The effect of the motion of rotation may be considered in this way: Let a rotating sphere be brought into contact with a plane slowly. The sphere will, of course, roll along the plane. The energy of translation and rotation being equal to the original energy, E, we shall have the same result in the case

under consideration, that is, we shall have the same velocity parallel to the plane, and the same angular velocity as if the sphere were in contact *with* the plane, because there is no slipping at the instance of contact.

Let v_1=velocity parallel to the plane. Then $\frac{v_1}{R}$=new angular velocity $=\omega_1$.

$$E=\tfrac{1}{5}MR^2\omega^2.$$

$$\text{Energy of translation}=\tfrac{1}{2}Mv_1^{\,2}.$$

$$\text{New energy of rotation}=\tfrac{1}{5}MR^2\omega_1^{\,2}=\tfrac{1}{5}Mv_1^{\,2}.$$

$$\therefore\ \tfrac{1}{5}MR^2\omega^2=\tfrac{1}{2}Mv_1^{\,2}+\tfrac{1}{5}Mv_1^{\,2}.\quad \text{Whence,}$$

$$v_1^{\,2}=\tfrac{2}{7}R^2\omega^2.$$

$$\therefore\ v_1=\sqrt{\tfrac{2}{7}}R\omega,\ \text{and}$$

$$\omega_1=\sqrt{\tfrac{2}{7}}\omega.$$

The distance which the sphere will move parallel to the plane while it is attaining its highest altitude will be $=tv_1=2\sqrt{\frac{a}{7g}}R\omega$.

From these data, knowing that the curve will be a parabola, we obtain

$$y^2=\frac{4R^2\omega^2}{7g}x,$$

the highest point in the origin. The distance between first and second impact is $4\sqrt{a/7g}\,R\omega$. As to the subsequent motion, we have the equation of energy

$$\tfrac{1}{2}Mv_1^{\,2}+\tfrac{1}{5}Mv_1^{\,2}=\tfrac{1}{2}Mv_2^{\,2}+\tfrac{1}{5}Mv_2^{\,2},\ \text{or}\ v_2=v$$

and the subsequent parabola will be the same as the first.

PROBLEMS.

37. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A thin board of which the elements are given is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A prolate spheriod of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

37. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Find the first four integral values of n in $\frac{n(5n-3)}{2}=\square$.

I. Solution by the PROPOSER, and Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

Let the heptagonal numbers $\frac{n(5n-3)}{2}=\square=y^2$. Clearing of fractions, then multiplying by 20 and adding 9 to both sides, $(10n-3)^2=40y^2+9=\square=x^2$. $\therefore n=(x+3)/10$........(1). Let $x^2-40y^2=9$ be written $3^2x_1{}^2-40.3^2y_1{}^2=3^2$. Dividing by 3^2 and solving $x_1{}^2-40y_1{}^2=1$, the convergent of $\sqrt{40}$ is $19/3$. $\therefore x_1=19$; by the general formula $x_{n+1}=2x_1\times x_n-x_{n-1}$, we have $x_1=1$, 19, 721, 27379, 1039681, 39, 480, 499, etc. As $x=3x_1$ and as integral values for n can only be obtained by the numbers ending in 9, then in (1) $n=1$, 6, 8214, and 11844150.

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

The expression readily reduces to $10n^2-6n=\square$......(1). It is readily seen that $n=1$ satisfies this equation. Take $n=m+1$, substitute it in (1), reduce and we have $10m^2+14m+4=\square=$(say) $(pm-2)^2$, from which we obtain $m=(4p+14)/(p^2-10)$. Take $p=4$ and we have $m=5$, and $n=6$, the second value. Now take $n=m+6$, substitute in (1) and reduce as before and we find, $m=43$, and $n=49$, the third value. In $(4p+14)/(p^2-10)$ take $p=19/6$, $p^2-10=1/36$ and we have $m=960$, and $n=961$, the fourth value.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

If we put the expression equal x^2 and reduce, we readily obtain $10n=3\pm\sqrt{40x^2+9}$. Putting $x=1$, 2, 9, 40 and 77, respectively, I find the first four integral values of n to be, respectively, ±1, 6, -25, and 49.

38. Proposed by H. C. WILKES, Skull Run, West Virginia.

Let n be any number and let $n^3+1=x$.
Then $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$. Demonstrate.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

The simplest way is to substitute the value of x and expand. An identity is the result.

II. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Substituting n^3+1 for x, $(n^3+1)^3+(2n^3-1)^3+(n^4-2n)^3=(n^4+n)^3$,

which, if we put c for n, is the same as equation (12) on page 155 of Vol. II., No. 9 of the *Mathematical Magazine*, and is an identity as will be found by performing the indicated operations and adding.

III. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

Suppose the statement true : $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$.

Then, $x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3=n^3x^3$.

Whence, $(x-1)(x^2-3x+3)-n^3(x^2-3x+3)=0$. Whence, $x-1-n^3=0$.

Whence, $n^3+1=x$, which is the hypothesis. $\therefore$ The above supposition is true.

IV. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tennessee.

Performing the operations we have

$$x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3, \text{ or}$$

$$9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3(x-1), \text{ but } n^3=x-1\ ;$$

hence $9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27(x-1)^2$, which upon reduction gives

$$9x^3-9x^2-9n^3x^2+n^3x^3, =9x^3-9x^2-9x^2(x-1)+n^3x^3,$$

$$=9x^3-9x^2-9x^3+9x^2+n^3x_3,\ =n_3x_3.$$

Also solved by *J. H. DRUMMOND, M. A. GRUBER, J. SCHEFFER*, and *G. B. M. ZERR*.

39. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The m^{th} root of the n^{th} power of an *integral* number is a perfect p^{th} power. What is the number ?

Solutions by J. H. DRUMMOND, LL. D., Portland, Maine ; M. A. GRUBER, A. M., Washington, D. C., and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $x^{1\div m}=a^p$, then $x=a^{pm}$, in which a may be any integral number : for the n^{th} power of a^{pm} must also be a p^{th} power. [J. H. DRUMMOND.]

Manifestly any whole number raised to the mp^{th} power.

[J. SCHEFFER.]

Let x=the integral number. Then $x^{n\div m}=a^p$. Raising to m^{th} power and extracting n^{th} root, we obtain $x=a^{mp\div n}$, or $\sqrt[n]{a^{mp}}$. $\therefore$ The required integral number is the n^{th} root of the mp^{th} power of any integer, mp being a multiple of n.

[M. A. GRUBER.]

Also solved by *G. B. M. ZERR*.

PROBLEMS.

45. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

46. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If any positive integral number N be divided by another positive integral number D, leaving a remainder of 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

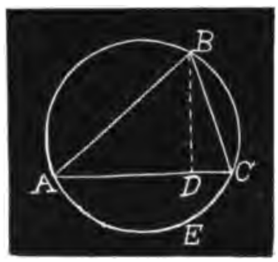
SOLUTIONS OF PROBLEMS.

28. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Missouri.

What is the average area of all triangles having a given base, b, and a given vertical angle, α?

Solution by the PROPOSER.

Let ABC be a triangle whose base $AC=b$ and vertical angle $ABC=\alpha$. Let $BC=x$, $\angle BAC=\theta$, and Δ average area required.



Then $x=\dfrac{b}{\sin\alpha}\sin\theta$, and $BD=x\sin\angle BCA$

$$=\frac{b}{\sin\alpha}\sin\theta\sin(\theta+\alpha).$$

$\therefore$ Area of the triangle $=\dfrac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)$.

The limits of θ are 0 and $\pi-\alpha$.

$$\therefore \Delta=\frac{\displaystyle\int_0^{\pi-\alpha}\frac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)d\theta}{\displaystyle\int_0^{\pi-\alpha}d\theta}=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\int_0^{\pi-\alpha}\sin\theta\sin(\theta+\alpha)d\theta$$

$$=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\Big[(-\tfrac{1}{2}\sin\theta\cos\theta+\tfrac{1}{2}\theta(\cos\alpha+\tfrac{1}{2}\sin^2\theta\sin\alpha\Big]_0^{\pi-\alpha}$$

$$=\frac{b^2}{4(\pi-\alpha)}\Big\{(\pi-\alpha)\cot\alpha+1\Big\}.$$

COROLLARY. Let $\alpha=\frac{1}{2}\pi$; then $\Delta=\dfrac{b^2}{2\pi}$, the same as problem 26.

[NOTE.—By mistake in numbering the problems in this department, number 28 was omitted. The above problem and solution are inserted that problems be numbered consecutively. EDITOR.]

29. Proposed by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

Neglecting perturbations, what is the average distance of the earth from the sun ?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy; Ohio University, Athens, Ohio.

The focus being the pole, the polar equation to the ellipse is

$$r=\frac{a(1-e^2)}{1-e\cos\theta}\dots\dots\dots\dots(1).$$

I. The radii vectores being drawn at equal angular intervals,

$$m'=\frac{\int r d\theta}{\int d\theta}=a(1-e^2)\frac{\int_0^\pi \frac{d\theta}{1-e\cos\theta}}{\int_0^\pi d\theta}=a\sqrt{1-e^2}=b.$$

II. If x be the abscissa of any point on the curve, the focal distance is

$$r=a-ex\dots\dots\dots\dots(2),$$

$$\text{and } m''=\frac{\int_{-a}^{+a}(a-ex)dx}{\int_{-a}^{+a}dx}=a,$$

the points on the curve being so taken that their abscissas increase uniformly.

III. If the number of radii vectores depends upon the length of the curve,

$$m'''=\frac{\int r ds}{\int ds},$$

ds being an element of the curve.

Also solved as I. above by *Profs. F. P. MATZ*, and *O. W. ANTHONY*, and as III. by *Prof. G. B. M. ZERR*.

PROBLEMS.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

38. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.**

Two arrows are sticking in a circular target: show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 335.]

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

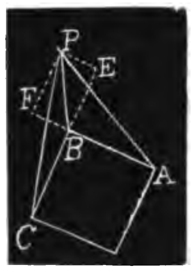
SOLUTIONS OF PROBLEMS.

33. **Proposed by Prof. ALEXANDER ROSS, C. E., Sebastopol, California.**

From a point P without a square field ABC, the distances PA, PB, and PC measured to the corners are, respectively, 70, 40, and 60 chains. What is the area of the field?

I. Solution by A. H. BELL, Hillsboro, Illinois, and A. H. HOLMES, Brunswick, Maine.

Let $a>b>c$ equal the distances 70, 60, and 40, and let $x=a$ side of the square field. Then $\cos A=\dfrac{a^2+x^2-c^2}{2ax}$, and this multiplied by $a=AF=\dfrac{a^2+x^2-c^2}{2x}$. $AF-AB=BF=EP=\dfrac{a^2-c^2-x^2}{2x}$; then, also, $BE=\dfrac{b^2-c^2-x^2}{2x}$.



$$\therefore \frac{(a^2-c^2-x^2)^2+(b^2-c^2-x^2)^2}{4x^2}=c^2 \ldots\ldots\ldots\ldots\ldots(1).$$

$$x^4-(a^2+b^2)x^2=c^2(a^2+b^2)-\frac{a^4+b^4+2c^4}{2} \ldots\ldots\ldots(2).$$

$$\text{Area of square}=x^2=\tfrac{1}{2}\left[a^2+b^2\pm\sqrt{4c^2(a^2+b^2-c^2)-(a^2-b^2)^2}\right] \ldots.(3).$$

Then area required$=(8500\pm6516.901)\div2=750.84\frac{1}{2}$ or 99.155 acres.

The second is the value required; the other is for point within the field.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let $ABCD$ be the square, $OA=70=a$, $OB=40=c$, $OC=60=b$, O the origin, (x, y) co-ordinates of A, (u, v) co-ordinates of C, $\angle ABE=\theta$, $\angle EBC=\frac{\pi}{2}-\theta$.

$$\therefore (x-c)^2+y^2=(u-c)^2+v^2,\ x^2+y^2=a^2,\ u^2+v^2=b^2 \ldots\ \ldots\ldots(1, 2, 3).$$

$\tan\theta=\dfrac{y}{x-c}$, $\cot\theta=\dfrac{v}{u-c}$, $\therefore\ \dfrac{y}{x-c}=\dfrac{u-c}{v}$. $\therefore\ yv=(x-c)(u-c)$..........(4.)

(2) and (3) in (1) and (4) gives, $2c(x-u)=a^2-b^2$, ...(5),

$(a^2-x^2)(b^2-u^2)=(x-c)^2(u-c)^2$.........................(6). (5) in (6) gives

$\{4a^2c^2-(a^2-b^2)^2-4cu(a^2-b^2)-4c^2u^2\}\ (b^2-u^2)=(a^2-b^2+2uc-2c^2)\ (u-c)^2$.

$\therefore\ (74175-520u-16u^2)(3600-u^2)-(4u-95)^2(u-40)^2$.

$\therefore\ 32u^3-2840u^2+825u+3157375=0$. Let $u=z+\frac{355}{12}$.

$\therefore\ z^3-\frac{249575}{96}z+\frac{1546768875}{3456}=0$.

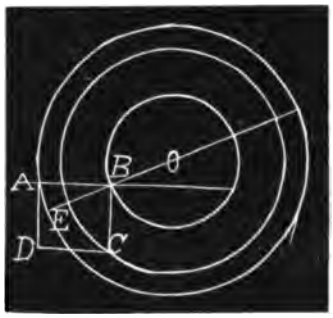
This equation has three roots.

$\therefore\ z_1=23.02208$, $z_2=36.23197$, $z_3=-58.43863$.

$\therefore\ u_1=52.60541$, $u_2=65.81530$, $u_3=-28.85530$.

$\therefore\ x_1=69.35541$, $x_2=82.56530$, $x_3=-12.10530$.

$\therefore\ y_1=9.47772$, $y_3=68.94535$.



The first values satisfy the problem in question; the second must be rejected as not admissible; while the third values satisfy the problem for the point within the field.

$\therefore$ area $ABCD=(x-c)^2+y^2=951.5672$ square chains $=95.15672$ acres.

When the point is within field, area $=(x-c)^2+y^2=7468.424$ square chains $=746.8424$ acres.

Also solved by *O. W. ANTHONY*.

34. Proposed by THOS. U. TAYLOR, C. E., M. C., Department of Engineering, University of Texas, Austin, Texas.

Given a variable parallelogrom $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and (C)$=h$, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by AP+area generated by BC).

No solution of this problem has been received.

PROBLEMS.

38. Proposed by F. M. PRIEST, Mona House, St. Louis, Missouri.

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 and 40 feet in height, are both standing perpendicular at the sea level. They start to fall in still air, how long will it require each one to fall to a horizontal position ?

39. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.**

A straight inflexible bar of uniform weight and thickness, length m is suspended at the two ends by a string without weight, length $l>m$ passing freely over a peg driven in a perpendicular wall. Describe and analyze the curve traced on the wall by the ends of the hanging bar.

NOTE.—Problem No. 43, Calculus, should read as suggested by Dr. E. A. Bowser on page 60 of February number. Prof. Black had noted the correct form in his copy of Williamson, and so sent it to the MONTHLY, but an error was made in printing the expression. A letter from Dr. Williamson, Trinity College, Dublin, Ireland, acknowledges the error in his work, and says it will be corrected in the forthcoming new edition of his book.

WANTED.—Some one to give a list, partial or complete, of the curves of the fourth degree that have received particular names, such as the "Lemniscate," "Cocked Hat," "Devil's Walking Stick," "Conchoid," etc.

COOPER D. SCHMITT.

BOOKS.

Warren Colburn's First Lessons: Intellectual Arithmetic upon the Inductive Method of Instruction (1891).

H. N. Wheeler's Second Lessons (1893). Boston, New York, and Chicago: Houghton, Mifflin and Company.

First Lessons, which has been famous for three-fourths of a century, contains, besides the four fundamental operations, little but fractions and denominate numbers. It has no rules and but little written work. It follows the inductive method—the method of "Practice before Theory"—which is based on the soundest psychological principles. This book should be in the hands of every teacher, whether used as a class book or not.

Wheeler's Second Lessons is intended as a continuation of Colburn's First Lessons, and is well adapted for that purpose. J. M. C.

Logarithmic Tables. By Professor George William Jones, of Cornell University. Sixth Edition. Royal 8vo. Cloth. 160 pages. Price, \$1.00. Published by the author.

These are the best tables that we have yet seen. Eighteen tables (four-place, six-place, ten-place) with full explanation for their use, for use in the class-room, laboratory, and the office. The tables of Mathematical Constants, Chemistry, Engineering, and Physics deserve special mention. Also Table IX which gives the prime factors of composite numbers less than 20000, and Tables X and XI which give the squares and cubes of all

three figure numbers in full. If you want a complete and valuable set of tables buy a copy of Prof. Jones, and you will need none other. B. F. F.

Mathematical Papers Read at the International Mathematical Congress held in Connection with the Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress, E. Hastings Moore, Oskar Bolza, Heinrich Maschke, Henry S. White. Large 8vo. Cloth, 412 pages. Price, $4.00, New York: Macmillan & Co.

This important collection of important mathematical papers is given to the mathematicians of all time at no small amount of labor at the hands of the editors.

It is especially fitting that these papers, many of which indicate the high-water mark of the development of mathematics at the present time, should be collected and bound for the benefit of the mathematicians of the centuries yet to be.

Neither the management of the Exposition nor the government of the United States had made any provisions for the publication of the proceedings of any of the Chicago Congresses. No publisher was found willing to issue the papers at his own risk.

At last a guarantee fund of one thousand dollars in all was subscribed, six hundred dollars by the American Mathematical Society, and four hundred dollars by members of the Society and other mathematicians. On the basis of this guaranty fund the publication of the volume of the papers was made possible, the American Mathematical Society assuming the financial, and the Chicago Committee the editorial responsibility. *Preface.* B. F. F.

NOTES.

Dr. William B. Smith, of the Tulane University of Louisiana, has in press the first volume of his Infinitisimal Analysis.

The June number of the MONTHLY will be mailed about the 16th of the month. In this issue will appear the biography of Mr. W. J. C. Miller.

Dr. George Bruce Halsted, of the University of Texas, and Dr. David E. Smith, of the Michigan State Normal School, will spend the summer in Europe. Dr. Halsted will visit Paris, Genoa, Buda Pest, Moskow, Kazan, etc.

ERRATA. In Prof. G. B. M. Zerr's paper, "The Centroid of Areas and Volumes," in value of $\bar{x}$, bottom of page 73, in numerator read $\frac{1}{4}(2p+1)$ for "$\frac{1}{4})2p+1)$," and in denominator read $\frac{k}{2}(2n+1)$ for "$\frac{k}{2}(2n=1)$" and $\frac{h}{2}(2m+1)$ for "$\frac{h}{3}(2m+1)$. Page 75, line 8 read $\left(\frac{z}{c}\right)$ for "$\frac{z}{c}$." Page 102, last line, read $-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)$ for "$-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)$," Page 103, first line, last expression in numerator read $\sqrt{2a^2h+4h^2}$ for "$\sqrt{2a^2h}+4h^2$" and in second line read dx for "bx."

W. J. C. MILLER.

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BIOGRAPHY.

MR. W. J. C. MILLER.

BY B. F. FINKEL.

THE native place of Mr. Miller is in one of the most beautiful parts of the South coast of England. Of this district he has given a sketch in an article (in Nature-Notes for August, 1894) entitled a "Devonian Headland," which he describes as lying deep within the great West Bay of Dorset and Devon, and to which sea-birds have always flocked as to a chosen retreat. The upland chalk downs end in lofty cliffs that run sheer to the water's edge: and close by, both east and west, clear brooks, which spring from the underlying green-sand, have worn out charming little valleys that bear the Celtic name of combes. The headland itself bears a Norse name, derived from a village that lies in the eastern valley,—it was a little way off on the shores of the same great bay, that the Norsemen had their first historic conflict with the English—but the village itself might well bear a similar place-name with its western neighbor, and be called more appropriately, Chalcombe.

The district was a perfect paradise of birds, with which he became perfectly familiar as a boy, and on which, in later life, he loved to write articles describing their various habits. The mobbing, by a mingled flock of rooks and jackdaws, of a pair of ancient ravens that had built for ages in a neighboring cliff, till, at last, the powerful ravens, worn out by numbers, would find shelter from their tormentors in some wood, or cleft, or cave: the motions and chirpings of the stone-chats and the win-chats in the furze-bushes: the swift and dazzling flight of the king-fisher: the finding of the habitat of the dipper or water-ouzel——a song-bird that dives, and wades, and swims—watching its motions under water, and finding its nest year after year in the same stream: and the delight-

ful turr-turr of the turtle-doves in the woods, reminding him that, for six months, he might say, with Virgil's Melibœus,

"Nec gemere aeria cessabit turtur ab ulmo":

these and many such sights and sounds he was ever delighted to recall and record.

In such pleasant regions as these, Mr. Miller was born, on August 31, 1832; here he roamed as a boy, always fond of books; and amid these scenes he acquired that love of Nature, and especially of bird-life, that never afterwards deserted him. From the village-school he went to the Independent College of Taunton in Somerset, where he had for a time, the teaching of a distinguished Mathematician; and from there he matriculated with mathematical honors at the University of London. Then came the great disappointment of his life. He was desirous to enter the great mathematical University of Cambridge; but his parents belonged to the sect that had trampled down King, Church, and Aristocracy, one after the other; that had formed an army that had never met, either in the British Islands or on the Continent, an enemy that could stand its onset; and had sent across the Atlantic a band which, fleeing from persecution, had founded the third home of the great English race. Thus they could not endure that a son of theirs should submit to the tests then imposed in the University; so he had to give it up. Years afterward, he learned from eminent mathematicians, that the best of all science was learned by one's own self, and never derived from any Professors at College or University. But he would then have gladly submitted to any test, if he had been allowed.

So he turned to study and instruction in mathematics; and after teaching at various Institutions, became finally Professor at Huddersfield College in Yorkshire, where he remained many years, till he took the post that he now holds. There it was that he devised, and, after many trials, got a Publisher to undertake, the series of Volumes that he has edited ever since, and of which he is now engaged upon the sixty-fifth Volume. It was in 1861 that he conceived the idea of devising some plan whereby the contributions to the mathematical columns of ths *Educational Times*, which had been for some years under his Editorship, might be presented, apart from other matter, in a more convenient form than could be furnished by the pages of the Journal; and, after ascertaining the views of his contributors, and obtaining promises of support, the mathematical solutions that appeared in each number were, from Midsummer, 1863, printed off, in the narrow columns then in use, from the journalistic types; and at the end of a year the collection was, in July, 1864, issued as the first of the series. By and by the narrow columns were altered to wider columns; and then the contributors were not content to wait a year for their articles: thus, ultimately, the issues took place at half-yearly intervals.

The series that took its rise from such small beginnings has gone on continuously from that time to this; and is going on still. After 25 years it was necessary to issue a second edition of the first volume; and this was brought out with improvements, uniform with the other Volumes, in the wider columns, in

1886. In these Volumes there have appeared, from time to time, articles in almost all branches of Mathematics, and the leading Mathematicians of all countries have continuously helped the work forward. One valued contributor, among early ones, was Dr. Hirst, F. R. S., who developed, in various articles, those elegant branches of Geometry in which all took a deep interest; and who, at last, collected and published his contributions in a separate Volume. Other important contributors to the early Volumes were Professor Cayley, to whom many articles were due; and the too-early lost Professor Clifford, who, being a fellow Devonian with the Editor, began to write when he was flying kites, and continued to furnish articles that increased in number and value through many volumes, accompanied by letters to the Editor that contained comments and developed views that were often more interesting than the articles themselves. The comparatively new theory of Local Probability was largely developed in the early Volumes by such writers as Woolhouse, Clarke, Crofton, Stephen Watson, our countryman, the late Professor E. B. Seitz who was a great master of difficult Probability Problems, and others. These contributors have all passed into the silent land. From a contributor who, it is to be feared, is getting near it, Professor Sylvester, articles followed in such quick succession that, from the very earliest times, there were but three or four numbers of the Journal, and those through the merest inadvertence, that did not contain, till the very last, at least one of his articles.

In 1876, Mr. Miller obtained the highly responsible post of Executive Officer (Registrar and Secretary) of the General Medical Council, an office in which he has remained ever since, continuing to edit, at his leisure, the mathematical periodical that has now attained to its 65th Volume. Among Editors of Mathematics this is deemed to establish for Mr. Miller what is termed "a Record": seeing that no other Mathematical Editor has ever, it is believed, gone on so long, with such laborious work as this. Always interested in Literature, no less than in Science, he edited for his Students at Huddersfield College a Magazine in which there came forward young contributors who afterward attained to eminence, whereof one has recently written an able book on the geography and resources of Africa.

During this time, he has been living in the finest of all the suburbs of London, in that Richmond whose name has been transferred to many other places, notably to that city which figured so largely in the Civil War. Under the title of "a Bird-loved Suburb of London" he has written an article to set forth its Bird-life, and its many beauties.

Here he founded in 1887, a Literary Society of which he is still President, and before which, on March 20th, one of his Mathematical contributors, Mr. George Heppel, M. A., lectured on the "Origins of European Poetry." In the course of his introductory remarks that evening, the chairman, Mr. Miller, said, "We are this evening entering upon a new departure. Hitherto, the lecturers have been members of our own society, but, in bringing in now for the first time a lecturer from outside, we are adopting a course that might hereafter

be worked out with advantage by our able and energetic secretary. Mr. Heppel is a mathematician, and such men have long been found peculiarly sensitive to the influence of the sister-arts of music and poetry. The very greatest of all living mathematicians [Professor Sylvester] called the attention of the Royal Society, twenty-five years ago, to the coincidence or parallelism, which observation has long made familiar, between the mathematical and the musical ethos : music being the mathematics of the sense, mathematics the music of reason ; the soul of each the same. Music the dream, mathematics the working life ; each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Beethoven-Gauss."

Other doings of Mr. Miller's during his life in Richmond, and his official duties at the General Medical Council, are set forth in the following article from the *Richmond and Twickenham Times* for August 17, 1889 :—

"Those who attended the meetings of the Richmond Athenæum, and the far larger number who read the reports of the proceedings of that body, are familiar with the pleasant, gracefully worded, and often erudite little speeches of Mr. W. J. C. Miller, a member of the council who has always been, in a double sense, a right-hand man to the chairman, sitting upon his right on the platform, and always ready, however abtruse the subject, to save a debate from flagging by filling up the regulation ten minutes with remarks which are always appropriate, often profound, invariably couched in the happiest words, and abounding in quotations from the poets, displaying a memory which is the admiration of all. Comparatively few, however, in Richmond know of the laborious and difficult duties in the world of mathematics to which Mr. Miller has devoted himself for more than thirty years, as editor of the *Educational Times*, or of the position which he has filled for thirteen years as the sole executive officer (registrar and secretary by name) in the management of the business of the General Medical Council.

With regard to Mr. Miller's editorial duties, many eminent mathematicians have given ungrudging testimony to their value. Thus Professor Sylvester—the first of living mathematicians—speaks of him as "an excellent mathematician, extensively and critically versed in all parts of the science, a good writer and lecturer on various subjects of natural science and other parts of human knowledge lying outside his own more special pursuits, and a most able and painstaking editor. . . . "His scientific attainments are of a high order; he is deeply skilled in nearly all the departments of the highest mathematics, and is a novice in none. His labour as mathematical editor of the *Educational Times*, in which his own original papers are fit company for those of our foremost analysts, is proof of that. It would be a mistake to suppose him a mere schoolmaster or a mere mathematician. He is a sound classical scholar, and an erudite man of letters." The late lamented Professor Clifford considered that the mathematical portion of the *Educational Times* "has done more to suggest and encourage original research than any other European periodical." Equally gratifying words are used by Sir Robert Ball, Professor Tait, Dr. Hirst, Rev. Dr. Booth, Professor Crofton, Colonel Clarke, Dr. S. T. Hall, Professor Townsend, Professor Young, Dr. Todhunter, Rev, George Salmon, Professor Cayley, Professor Everett, and others whose attainments have raised them to the highest eminence. It has often been said that by Mr. Miller's mathematical work, the culture and study of the science have been more advanced than by any two or three agencies put together, in any or all parts of the world. When he commenced this important work he had but what was then an utterly obscure and almost unknown journal to use as his means of intercommunication and publication. Now he has nearly five hundred vigorous contributors, from all parts of the globe. Many are educated Hindoos (professors and others); many are Americans or Australians; still more are Germans, Frenchmen, Russians, or Italians; some are Spaniards; and some write from the South American Republics.

The multifarious work of the General Medical Council has more than quadrupled since Mr. Miller took it in charge thirteen years ago. Established to carry out the voluminous Medical Acts (which cover fifty-nine pages of the *Medical Register*), the Council had to take charge, in 1878, of all the dentists in the empire, and since then of various other matters, including, quite recently, the registration of sanitary officers. Many testimonies to the appreciation of Mr. Miller's services have been given by the Council, and by the medical newspapers. Thus the *Medical Press*, in a recent article on the General Medical Council, says that—"Every session marks a distinct improvement in the business aptitude of the Council, and in the amount of work accomplished, results which may fairly be attributed in no small degree to a more vigorous presidential control, and to the efficiency of the business arrangements, which depend so much for their success on the services of a competent and attentive registrar." The *British Medical Journal*, in reviewing the *Minutes of the General Medical Council*, says—"The Volume has been edited by Mr. W. J. C. Miller, B. A., the Registrar of the General Council, with the care which he has accustomed us to expect from him." The *Report of the Statistical Committee of the General Medical*

Council is another work of which Mr. Miller has charge, and in noticing this the *Medical Press* says—"We assume that its compilation is chiefly due to the energy and noted mathematical skill of Mr. Miller, the Registrar of the Council, and if we are correct in this assumption we can only remark that both the profession and the Medical Council owe that gentleman much thanks for work which, though no doubt a labour of love, must involve great devotion of time and mental capacity." Another work of the utmost importance to the public, and for the annual publication of which Mr. Miller is responsible, is the *Medical Register*, which has now grown to a volume of 1,198 pages. In addition to this there is the *Dentist's Register* (282 pages), besides the *Medical Students' Register*, the latter alone requiring 100 pages. It would be difficult to speak too highly of the care exhibited in the compilation of these important works. Referring to the issues for the present year, the *Medical Press* says "They display all the progressive improvement which has been manifested since Mr. Miller took them in hand."

Being an ardent lover of science and literature, Mr. Miller has all his life striven to aid others in sharing their delights, by lectures, writings, and teaching. And all this work, editorial and other, has been not only unremunerative, but carried on with no little outlay. But the world's best workers have always been the most unselfish. Mr. Miller has at least the gratification of knowing that his favourite pursuits have been greatly advanced by his efforts, and that he has earned the gratitude of many who have reaped the advantages of his self-denying work."

Mr. Miller was one of the earliest members of the London Mathematical Society ; but as he found that, with his official duties, and his Editorial work, he could not spare time to attend the Meetings, he was reluctantly compelled to resign his membership. Since that time, he has had to devote the whole of his small leisure to the duties of his Editorship, which goes on increasing every month, with new contributors from foreign countries, especially India, where an enlarged interest is rapidly growing in all the articles that are pnblished in his Journal. Mr. Miller is a great admirer of America and American ways of managing ; he entertains a high opinion of our magazine, and says it is one of the best that comes to him. He has a large circle of friends and admirers in America, most of whom are contributors to the Mathematical Department in the *Educational Times*.

THE EXPONENTAL DEVELOPMENT FOR REAL EXPONENTS.

By WILLIAM BENJAMIN SMITH. Ph. D., (Görtingen) Professor of Mathematics, Tulane University, New Orleans, Louisiana.

The Exponental Series is of such fundamental and far-reaching importance, it is so indispensable to all higher Analysis, that it seems strange so few if any deductions of it accessible to the English reader should be carefully conducted ; not even that given by Chrystal in his superb *Treatise on Algebra* can lay claim to rigor. It may be worth while then, under no pretense of novelty, to attempt to supply this lack in some measure.

I. We consider the expansion given by the Binomial Theorem :

$$\left(1+\frac{x}{n}\right)^n=1+n.\frac{x}{n}+\frac{n(n-1)}{\underline{|2}}.\frac{x^2}{n^2}+\frac{n(n-1)(n-2)}{\underline{|3}}.\frac{x^3}{n^3}+\ \ldots\ldots$$

$$=1+x+(1-\frac{1}{n}).\frac{x^2}{\underline{|2}}+(1-\frac{1}{n})(1-\frac{2}{n}).\frac{x^3}{\underline{|3}}+\ldots\ldots ,$$

where x is finite and positive, while n is *positive* and *integral*, and we inquire whether this series tends toward a definite form and value as n increases without limit.

We denote the differences $1-\frac{1}{n}$, $1-\frac{2}{n}$, ... $1-\frac{k}{n}$ ..... by d_1, d_2, ...d_k, ... and the products of these, d_1, d_1d_2, $d_1d_2d_3$, ...d_1d_2 ......d_k, ...by p_1, p_2, p_3, ...p_k, ...

Then plainly $1>d_1>d_2>d_3>\ldots\ldots>d_k>\ldots\ldots$

and also $1>p_1>p_2>p_3>\ldots\ldots>p_k>\ldots\ldots$

In the expansion there are $n+1$ terms, which we may write t_0, t_1, t_2, ...t_r, ...t_n. We consider the sum $t_0+t_1+\ldots\ldots+t_r$ and denote it by S_r; then the sum of remaining $n-r$ terms we denote by V_r, so that $\left(1+\frac{x}{n}\right)^n=S_r+V_r$ where

$$S_r=1+x+p_1\frac{x^2}{\underline{|2}}+p_2\frac{x^3}{\underline{|3}}+\ldots\ldots+p_k.\frac{x^{k+1}}{\underline{|k+1}}+\ldots\ldots+p_{r-1}.\frac{x^r}{\underline{|r}},$$

$$V_r=p_r.\frac{x^{r+1}}{\underline{|r+1}}+p_{r+1}.\frac{x^{r+2}}{\underline{|r+2}}+\ldots\ldots+p_{n-1}.\frac{x^n}{\underline{|n}}.$$

Since n is to be taken *great at will*, r may also be taken *great at will* and yet *always less than n*. We now ask, what becomes of S_r as r increases without limit while always $r<n$? Since all the p's are <1, it is plain that

$$S_r<\{1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ \ldots\ldots+\frac{x^r}{\underline{|r}}\}.$$

However we can make *each* of the p's$>1-\sigma$, where σ is *small at will*. It is enough to prove this for the *least* of the p's, p_r. We have

$$p_r=(1-\frac{1}{n})(1-\frac{2}{n})\ldots\ldots(1-\frac{r-1}{n})>(1-\frac{r-1}{n})^{r-1}.$$

Now $(1-\frac{r-1}{n})^{r-1}=1-\sigma$ if $1-\frac{r-1}{n}=(1-\sigma)^{\frac{1}{r-1}}$, or if $\frac{r-1}{n}=1-(1-\sigma)^{\frac{1}{r-1}}$, or if

$$n \overline{\gtrless} \frac{r-1}{1-(1-\sigma)^{\frac{1}{r-1}}}.$$

Now for any finite value of r *however great*, and for any finite value of σ *however small*, this fraction on the right will always be finite and perfectly difinite, though perhaps very great; hence it will always be possible to choose n equal or even greater (in case the fraction be not integral in value); hence it will always be possible to make $p_r > 1-\sigma$, no matter how great r or how small σ, by merely choosing n great enough, and this we can always do, since n is quite at our will.

$$\text{Hence } S_r > (1-\sigma)\{(1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

$$\text{Hence } \{1+x+\frac{x^2}{\underline{|2}}+\ldots\ldots+\frac{x^r}{\underline{|r}}\} > S_r > (1-\sigma)\ \{1+x+\frac{x^2}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

Hence S_r differs from $\{\ldots.\}$ by less than $\sigma\{\ldots.\}$. Now this brace, $\{\ldots.\}$ is of course finite for all finite values of r, and it also remains finite (for x finite) even for r increasing without limit. For the ratio of two consecutive terms is $\frac{x}{k}$ and this ratio is not only finitely <1 for $k>x$ but it becomes ever smaller and smaller, sinking below every assignable degree of parvitude as k increases without limit, x of course being finite and fixed, however great. Hence $\sigma\{\ldots\ldots\}$ is small at will, since the product of a magnitude small at will multiplied by a finite number is itself small at will. Hence the two magnitudes $\{\ \ldots..\}$ and $(1-\sigma)\{\ldots..\ \}$ close down upon each other as r increases without limit, hence they close down upon S_r always between them, so that we have

$$\text{Lim.} S_r = 1+x+\frac{x^r}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}$$

for r and n increasing without limit, $n>r$. It remains to examine V_r. We may

$$\text{write } V_r = t_{r+1}(1+d_{r+1}\frac{x}{r+2}+d_{r+1}d_{r+2}\frac{x^2}{(r+2)(r+3)}+\ \ldots.\ ).$$

Now $t_{r+1} < \frac{x^{r+1}}{\underline{|r+1}}$ and this we have just seen is small at will for r great at will, or $t_{r+1} < \sigma$. Also the parenthesis $(\ \ldots.\ ) < [1+\frac{x}{r+2}+\frac{x^2}{(r+2)^2}+\ldots.]$,

and this bracket $[\ldots\ldots]$ is finite for all finite values of n and r, and it has 1 for its limit as n and r increase without limit. Hence $V_r < \sigma$, or Lim. $V_r = 0$, hence

$$\text{Lim.}\,(1+\frac{x}{n})^n=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots \textit{ in infinitum.}$$

II. Thus far n has been integral at every stage of value. What if it be *fractional* or *irrational?* The preceding proof does not then apply but we shall always have n lying between two consecutive integers, or $w<n<w+1$.

Now $(1+\frac{x}{n})^n<(1+\frac{x}{w})^{w+1}$, a greater number raised to a higher power; or $(1+\frac{x}{n})^n<(1+\frac{x}{w})^w\,(1+\frac{x}{w})$. Likewise $(1+\frac{x}{n})^n>(1+\frac{x}{w+1})^w$, a smaller number to a lower power. Or $(1+\frac{x}{n})^n>(1+\frac{x}{w+1})^{w+1}/(1+\frac{x}{w+1})$.

Now for w and $w+1$ increasing without limit we have just proved that $(1+\frac{x}{w})^w$ and $(1+\frac{x}{w+1})^{w+1}$ both close down upon one and the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

Also the multiplier $1+\frac{x}{w}$ and the divisor $1+\frac{x}{w+1}$ both close down upon the same limit 1; hence the product $(1+\frac{x}{w})^w.(1+\frac{x}{w})$ and the quotient $(1+\frac{x}{w+1})^{w+1}/(1+\frac{x}{w+1})$ both close down upon the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots;$$

hence $(1+\frac{x}{n})^n$ lying always between this product and this quotient itself closes down upon the same limit; hence $\text{Lim.}(1+\frac{x}{n})^n=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ for finite positive x and for positive n increasing no matter how without limit.

III. For x *negative* we must consider $(1-\frac{x}{n})^n$. This we write $=E-O$, a sum of even powers, less a sum of odd powers, of x. Each of these we break up into two parts, S_r and V_r, and reiterate the foregoing argument with insignificant and self-evident modifications. There results

$$\text{Lim.}(1-\frac{x}{n})^n = 1-x+\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\ldots\ldots$$

for finite positive x and n increasing no matter how without limit. Or

$$\text{Lim.}(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$$

for any finite real x positive or negative, n increasing any way without limit.

IV. For n *negative* we have

$$\left(1-\frac{x}{n}\right)^{-n}=\left(\frac{n-x}{n}\right)^{-n}=\left(\frac{n}{n-x}\right)^n=\left(1+\frac{x}{n-x}\right)^n=$$

$$\left(1+\frac{x}{n-x}\right)^{n-x}\cdot\left(1+\frac{x}{n-x}\right)^x=\left(1+\frac{x}{n-x}\right)^{n-x}\Big/\left(1-\frac{x}{n}\right)^x.$$

Now as n increases without limit, so also does $n-x$, for x finite no matter how great; hence $(1+\frac{x}{n-x})^{n-x}$ approaches as its limit the series $1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$ and $(1-\frac{x}{n})^x$ approaches 1 as its limit manifestly; hence

$$\text{Lim.}(1-\frac{x}{n})^{-n} = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$$

Hence $\text{Lim.}(1+\frac{x}{n})^n=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$ *for all finite real values of x, for real n increasing without limit no matter how, positively or negatively.*

For $x=1$ we obtain $\text{Lim.}(1+\frac{1}{n})^n=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\ldots.$

The number defined by this series and denoted by e, is one of the three irrationals (π, i, e) all-important to analysis. That e is irrational may be easily seen thus: Consider the first $(p+1)$ terms $1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\ldots\frac{1}{\lfloor p}$; the sum Σp is $\frac{w}{\lfloor p}$ where w is some integer no matter what; the remainder

$$\frac{1}{\lfloor p+1}+\frac{1}{\lfloor p+2}+\ldots\ldots \text{ is } \frac{1}{\lfloor p}\left\{\frac{1}{p+1}+\frac{1}{(p+1)(p+2)}+\ldots\ldots\right\}$$

$$< \frac{1}{\underline{|p}}\left(\frac{1}{p+1} = \frac{1}{(p+1)^2} + \ldots\ldots\right) < \frac{1}{\underline{|p}}. \quad \text{Hence} \frac{w}{\underline{|p}} < e < \frac{w+1}{\underline{|p}}.$$

Now as p increases without limit these two fractions close down upon each other and upon e always between them. It is plain that there is no fixed fraction as N/D, always between $\frac{w}{\underline{|p}}$ and $\frac{w+1}{\underline{|p}}$; for however great D might be, we could choose p so large, that $\underline{|p}$ would include all the factors of D; hence $\frac{N}{D} = \frac{w}{\underline{|p}}$, whereas $e > \frac{w}{\underline{|p}}$. In fact as the two fractions $\frac{w}{\underline{|p}}$ and $\frac{w+1}{\underline{|p}}$ close down on each other and on their common limit e they *pass over* (either the one or the other) every assignable fraction lying between them.

The importance of e lies in the fact that the series $1+x+\frac{x^2}{\underline{|2}}+\ldots\ldots$ is expressible through it. We have

$$\left(1+\frac{x}{n}\right)^n = \left\{\left(1+\frac{x}{n}\right)^{n/x}\right\}^x = \left\{\left(1+\frac{1}{n/x}\right)^{n/x}\right\}^x. \quad \text{Hence}$$

$$\text{Lim.}\left(1+\frac{x}{n}\right)^n = \text{Lim.}\left\{\left(1+\frac{1}{n/x}\right)^{n/x}\right\}^x = \left\{\text{Lim.}\left(1+\frac{1}{n/x}\right)^{n/x}\right\}^x,$$

where we indeed assume that the Limit of the Power equals the Power of the Limit. But this is plainly correct, at least in the present case; for

$$\left(1+\frac{1}{n/x}\right)^{n/x} = e+\sigma.$$

Hence $\text{Lim.}\left\{\left(1+\frac{1}{n/x}\right)^{n/x}\right\}^x = \text{Lim.}(e+\sigma)^x = \text{Lim.}(e^x+\sigma x e^{x-1}+\ldots\ldots) = e^x$ for all finite real values of x. Hence

$$\text{Lim.}\left(1+\frac{x}{n}\right)^n = e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots \textit{ in infinitum.}$$

Herewith then the exponental development is established for all finite real values of the exponent.

Tulane University, May, 1896.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from April Number.]

VII. Let ABC be a $\triangle$ right angled at C. Produce BC making $BD=BA$. Join DA. From E, the middle point of CD, draw a perpendicular meeting DA, as at F. Join FB. $\triangle ADC$ is similar to $\triangle BFE$.

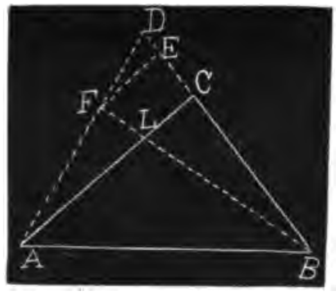


Fig. 7.

$\therefore AC : BE :: DC : FE$.

$\therefore AC : BC+(AB-BC)\div 2 :: AB-BC : AC\div 2$.

$\therefore \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

NOTE.—This proof is credited to Hoffman.

VIII. Triangles BDF, BFE, and FDE, are similar.

Letting $BD=BA=c$, $AC=b$, $BC=a$, $BE=\frac{a+c}{2}$, $DE=\frac{c-a}{2}$, $FE=\frac{b}{2}$, $DF=x$, $BF=y$, we obtain the following:

(1). $\frac{c-a}{2} : x :: x : c$. $\therefore x^2=\frac{c(c-a)}{2}$ .... 1.

(2). $\frac{c-a}{2} : x :: \frac{b}{2} : y$. $\therefore bx=(c-a)y$ .... 2.

(3). $x : c :: \frac{b}{2} : y$. $\therefore xy=\frac{bc}{2}$ .... 3.

(4). $\frac{c-a}{2} : \frac{b}{2} :: \frac{b}{2} : \frac{c+a}{2}$. $\therefore c^2-a^2=b^2$ .... 4.

(5). $\frac{c-a}{2} : \frac{b}{2} :: x : y$. $\therefore bx=(c-a)y$ .... 2.

(6). $\frac{b}{2} : \frac{c+a}{2} :: x : y$. $\therefore by=(c+a)x$ .... 5.

(7). $\frac{c+a}{2} : y :: y : c$. $\therefore y^2=\frac{c(c+a)}{2}$ .... 6.

(8). $\frac{c+a}{2} : y :: \frac{b}{2} : x$. $\therefore by=(c+a)x$ .... 5.

(9). $y : c :: \frac{b}{2} : x$. $\therefore xy=\frac{bc}{2}$ .... 3.

From 4 we get $c^2=a^2+b^2$.

The set 2 and 5 gives the same result. But equation 2 may come from proportion (2) or (5), and 5 from (6) or (8), thus making four proofs for this set.

The following sets of three equations furnish fourteen proofs, since each set can come from two or more sets of three proportions: 1, 2, 6; 1, 3, 5; 1, 3, 6; 1, 5, 6; 2, 3, 6. Total number of proofs for this method is 19.

IX. Comparing the triangles BDF, BFE, ADC, BLC, and ALF, Fig. 7, we may put the result in the following condensed form:

$$DF=x : EF=\tfrac{1}{2}b : DC=c-a : LC=z : FL=y-v$$

$$:: FB=y : EB=\tfrac{1}{2}(c+a) : AC=b : CB=a : AF=x$$

$$:: BD=c : FB=y : AD=2x\ LB=v : AL=b-z.$$

From this we easily may derive thirty different simple proportions, which give twenty-seven different equations. Some idea of the number of proofs that may be obtained from different sets of these equations, can be formed from the fact that there are 17550 sets of four equations, to say nothing of sets of three and of five. Of course, many of the sets must be rejected for reasons stated fully in V. We leave details to the reader.

X. Suppose the theorem true. Then $\overline{AB}^2=\overline{AC}^2+\overline{BC}^2$, $\overline{BC}^2=\overline{CD}^2+\overline{BD}^2$, and $\overline{AC}^2=\overline{AD}^2=\overline{CD}^2$.

Fig. 1.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2\overline{CD}^2+\overline{BD}^2$.

But $\overline{CD}^2=AD\cdot BD$.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2AD\cdot BD+\overline{BD}^2$.

$\therefore\ AB=AD+BD$, which is true.

$\therefore$ The supposition is true.

NOTE.—This method is credited to Hoffman.

XI. In Fig. 1, $\overline{AB}^2<,=,\text{ or }>\overline{AC}^2+\overline{BC}^2$. Suppose it less. Then, since $\overline{AB}^2=(AD+DB)^2=(\overline{CD}^2\div DB+DB)^2$, and $\overline{AC}^2=(\overline{CD\cdot BC}\div DB)^2$,

$$(CD^2\div DB+DB)^2<(\overline{CD\cdot BC}\div DB)^2+\overline{BC}^2.$$

$\therefore\ (\overline{CD}^2+\overline{DB}^2)^2<\overline{BC}^2(\overline{CD}^2+\overline{DB}^2)$.

$\therefore\ \overline{BC}^2>\overline{CD}^2+\overline{DB}^2$, which is absurd. For were the supposition true, we should have $\overline{BC}^2<\overline{CD}^2+\overline{DB}^2$, as can easily be shown.

Similarly the supposition that $\overline{AB}^2>\overline{AC}^2+\overline{BC}^2$ can be proven false.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XII. $BC=a : EF=x : DF=y : DE=z$

$:: AC=b : AF=v : EF=x : AE=w$

$:: AB=c : AE=w : ED=z : AD=v+y$.

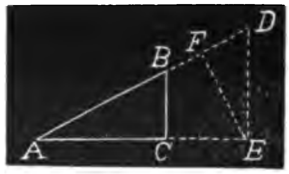


Fig. 8.

The above condensed form is self-explanatory, as are also the two following.

We leave the selection of simple proportions, the derivation and solution of consequent equations, as an exercise for the interested reader.

XIII. $BC=a : DE=x : DL=y$,

$:: AB=b : AE=z : LF=FE=v$,

$:: AB=c : AD=v+y : DF=x-v$.

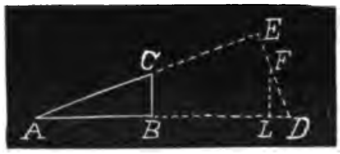


Fig. 9.

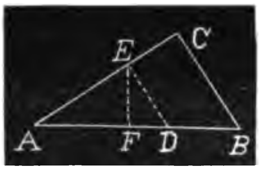


Fig. 10.

XIV. $BC=a : ED=EC=x : FD=y : EF=z$,

$:: AC=b : AE=b-x : EF=z : AF=v$,

$:: AB=c : AD=v+y : ED=x : AE=b-x$.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from May Number.]

Each of the regular groups of degree six contains only one subgroup of the type $(abc.def)$. Since no substitution of the form $abcde$ can transform this into itself the group of order 30 is impossible.

If a group of order 60 exists it must contain six subgroups of the type $(abcde)_{10}$. We may assume that it contains $(abcde)_{10} \equiv G_f$. G_b must then contain $ac.de$ and a substitution of the type $abcde$ which contains the letters a, c, d, e, f. We may assume that this substitution is $acd_1e_1f_1$. It is then necessary that $ac.de.acd_1e_1f_1 = af_1e_1d_1c.ac.de$. Hence

$$acd_1e_1f_1 = acdfe \text{ or } acefd.$$

Since *acefd.adbec*=*bef* every group of order 60 must contain

$$(abcde)_{10} \text{ and } acdfe.$$

These substitutions generate a group whose order $\overline{>}60$, hence only one group of order 60 is possible.

We shall prove that these substitutions generate a group of order 60 by employing a very elementary but somewhat lengthy method. Representing the substitutions of $(abcde)_{10} \equiv 1$, *abcde*, *acebd*, *adbec*, *aedcb*, *ab.ce*, *ac.de*, *ad.bc*, *ae.bd*, *be.cd* respectively by $1=s_1$, s_2, s_3, s_4, s_5, c_6, s_7, s_8, s_9, s_{10} and *acdfe* by t, we form the rectangle

$$\begin{array}{llll}
1 & s_2 & s_3 & \ldots\ldots\ldots\ldots s_{10} \\
t & s_2 t & s_3 t & \ldots\ldots\ldots\ldots s_{10} t \\
t^2 & s_2 t^2 & s_3 t^2 & \ldots\ldots\ldots\ldots s_{10} t^2 \\
t^3 & s_2 t^3 & s_3 t^3 & \ldots\ldots\ldots\ldots s_{10} t^3 \\
t^4 & s_2 t^4 & s_3 t^4 & \ldots\ldots\ldots\ldots s_{10} t^4 \\
t_1 & s_2 t_1 & s_3 t_1 & \ldots\ldots\ldots\ldots s_{10} t_1
\end{array}$$

Where t_1 is any substitution generated by $(abcde)_{10}$ and *acdfe* which is not found in the preceding five rows. These substitutions are all different. They form a group if $t_1{}^2$ is contained in the first five rows and

$$t^{\alpha} s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1, \; t_1 s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1$$

$$(\beta, \gamma = 1, 2, \ldots\ldots 10), (\alpha, \delta = 1, 2, \ldots\ldots 4).$$

Instead of allowing β to have 10 values it is clearly sufficient to assign to it only the two values of 2 and 6 since *abcde* and *ab.ce* generate $(abcde)_{10}$. The following shows that the necessary conditions are fulfilled:

$$\begin{array}{ll}
t s_2 = adf.bce = s_{10} t^2 & t s_6 = aeb.cdf = s_3 t^3 \\
t^2 s_2 = aed.bcf = t_1{}^* & t^2 s_6 = adcfb = s_9 t_1 \\
t^3 s_2 = afdbc = s_9 t^4 & t^3 s_6 = afedb = s_4 t \\
t^4 s_2 = bc.ef = s_5 t & t^4 s_6 = acb.def = s_8 t^4 \\
t_1{}^2 = ade.bfc = s_5 t^3 & t_1 s_2 = bd.cf = s_4 t^3 \\
 & t_1 s_3 = acf.bed = s_9 t^2
\end{array}$$

There is therefore one group of order 60, viz:

(1) $$(abcde)_{10}(acdfe) = (abcdef)_{60}.$$

If there is a primitive group of order 120 it may be assumed that it con-

*In the above rectangle f is followed by the same letter as in the corresponding t or t_1. Since it is not followed by b in t, *aed.bcf* cannot be contained in the first five lines and may therefore be used for t. All these relations may be readily found if this property is observed.

tains $(abcde)_{20}$ and therefore $(abcdef)_{60}$. Since half of its substitutions must be negative it must contain $(abcdef)_{60}$ as a self-conjugate subgroup.

The order of a group which satisfies these conditions cannot be less than 120. From this we see that there cannot be more than one group of this order. That there is one follows from the facts that *acbe* belongs to $(abcde)_{20}$ and transforms *acdfe* into $acbdf=(s_0t_1)^2$=some substitution of $(abcdef)_{60}$.

The other primitive groups of degree six must contain subgroups of degree five which contain substitutions of one of the two types

ab *abc*

They must therefore be the alternating and the symmetric group. The following is therefore a complete list of these groups:

Order	Group
60	$(abcdef)_{60}$
120	$(abcdef)_{120}$
360	$(abcdef)$pos
720	$(abcdef)$*all*

REMARKS.

We have now finished the explanations of the elementary methods of group construction. By means of these we have been able to find, with a reasonable amount of labor, all the groups whose degree does not exceed six. It scarcely needs to be stated that this labor could have been considerably reduced by employing more advanced methods. In fact, we did not endeavor so much to find these groups by the least labor as to find them in such a way as to illustrate some of the most important elementary methods of group construction.

We are indebted to our honored teacher, Professor F. N. Cole not only for many of these methods but also for the fundamental ideas.

Most of the theorems that we have developed are found in Part I of Netto's Theory of Substitutions (American Edition). In some instances it seemed desirable to change the method of proof either because we had not yet developed the principles upon which Netto's proof is based, or because we desired to call attention to some special property. In a few instances our purposes required us to pursue the demonstration farther than is done by this author.

We did not enter into a special study of methods of operating with substitutions. Some of the more important ones have been incidentally explained. For further explanations we would refer to Senet's Algèbre Supérieure, Part IV, (this part is found in the second volume of this work), and to Part I of Netto.

In these works is also found considerable on the analysis of a substitution. The first 15 pages of the first volume of Gordan's Invariantentheorie contain considerable on this point. For the more advanced methods of operation we have

to refer to the classical work on this subject, Jordan's Traité des Substitutions, and to the periodicals.

Before entering upon the development of more advanced methods of group construction we shall study some of the relations which exist between substitution groups and functions containing a finite number of letters. These relations will not only show how substitution groups may be utilized but they may also serve as a means of arriving at important properties of substitution groups.

Leipzig, Germany, September 20, 1895.

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, M. A., Instructor of Mathematics, Waco High School, Waco, Texas.

[Continued from May Number.]

The discussion in this article is restricted to two unknown quantities. Cases 1, 2, 4, and 5 apply to two variables just as they are stated in the previous article. In Case 3, the restriction that each factor must occur twice is unnecessary when only two variables occur. It is sufficient for one factor to occur in each equation. This reduces Case 3 to Case 2. For two variables, Cases 6, 7, and 8 become one and the same, as no restrictions are necessary.

The following Cases are applicable to two variables only. Express the equations thus for Cases 9 and 10:

$$ax^2+by^2+cxy+dx+ey+f=0. \qquad 1.$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0. \qquad 2.$$

CASE 9. When $a:a'::c:c'::d:d'$. If this is true, it is obvious that the terms containing x can be eliminated. This holds true when the terms of any one, or any two, of the three ratios are zero.

CASE 10. When $a:a'::b:b'::d^2:d'^2::e^2:e'^2$, and when $d:d'::e:e'$.

By alternation, $\dfrac{e}{d}=\dfrac{e'}{d'}$, $\dfrac{b}{a}=\dfrac{b'}{a'}$.

Let $\dfrac{e}{d}=r$. Then $\dfrac{e'}{d'}=r$, $\dfrac{b}{a}=r^2$, $\dfrac{b'}{a'}=r^2$.

$e=dr,\ e'=d'r,\ b=ar^2,\ b'=a'r^2.$

Divide equation 1 by ad, and equation 2 by $a'd'$.

$$\frac{x^2+r^2y^2}{d}+\frac{x+ry}{a}+\frac{cxy+f}{ad}=0. \qquad \frac{x^2+r^2y^2}{d'}+\frac{x+ry}{a'}+\frac{c'xy+f^2}{a'd'}=0.$$

Let $x=r(u+v)$, $y=u-v$, and substitute. Eliminate v^2 and solve the resulting equation. The signs of d and e, and of d' and e' must be alike in both, or unlike in both equations. They cannot be alike in one and unlike in the other. Symmetrical equations are a special form of this case.

CASE 11. When the two equations can be expressed as follows:

$$(mx+p)^2-(m'x+p')^2+r(ny+q+n'y+q')=0$$

$$(ny+q)^2-(n'y+q')^2+r'(mx+p+m'x+p')=0.$$

By factoring, one value, and only one, of x and y can be found. The "Harvard Catch" is a special form of this Case.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

59. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me $1\frac{1}{2}$ per cent. brokerage for buying some uncurrent bank bills at 20 per cent. discount. Of these bills, 4 of \$50. each become worthless, but the remainder I dispose of at par, and make by the operation \$364. What was the face amount ? [Which answer is correct, \$3000, or \$$3048\frac{24}{37}$?]

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas; H. C. WILKES, Skull Run, West Virginia; and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

$80\%+1\frac{1}{2}\%=81\frac{1}{2}\%$; $100\%-81\frac{1}{2}\%=18\frac{1}{2}\%$.

\$364+(4×\$50)=\$564, amount he would have made if he had disposed of all at par.

\$564÷$18\frac{1}{2}\%$=\$$3048\frac{24}{37}$, the correct face amount.

The other result is obtained as follows:

$1\frac{1}{2}\%$ of $80\%=1\frac{1}{5}\%$.

$80\%+1\frac{1}{5}\%=81\frac{1}{5}\%$; $100\%-81\frac{1}{5}\%=18\frac{4}{5}\%$.

$\$564\div18\frac{4}{5}\%=\$3000$.

The latter is commission on money invested and not brokerage on bills bought.

II. Solution by F. M. McGAW, Bordentown, New Jersey.

Market Value+Brokerage equals whole cost, therefore gain % was $1.00-(.80+.015)=.185$.

The net gain in money was \$364 to which we add the \$200 lost, making a gross gain of \$564. Then $18.5\%=\$564$, whence $\$564\div.185=\$3058\frac{24}{37}$, face.

To determine which answer is correct, assume the answer and work backwards.

I. Assume $\$3048\frac{24}{37}$ as face, then

$\$3048\frac{24}{37}\times.815=\text{cost}=\$2484\frac{24}{37}$,

$\$3048\frac{24}{37}-\$200(\text{lost})=\$2848\frac{24}{37}$

$\$2848\frac{24}{37}-\$2484\frac{24}{37}=\$364$, net gain. Answer.

II. Assume \$3000 as face, then the same operations produce a gain of only \$355.

Also solved by *A. P. REED*, *H. C. WHITTAKER*, *P. S. BERG*, and *J. SCHEFFER*.

We received solutions of problem 58, too late for credit in last issue, from *J. SCHEFFER*, *E. R. ROBBINS*, and *P. S. BERG*.

PROBLEMS.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60$ per cent.?

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6 per cent.?

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

[77 2-9 rods and 88 8-9 rods have been obtained. Which is correct?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running?

I. Solution by FREDERICK R. HONEY, A. B., New Haven, Connecticut.

The problem is independent of the number of poles counted, and of the number of miles per hour the train is running. We will call this number N.

$\therefore aN=$the number of yards the train runs while the poles are counted. Also, $1760N=$number of yards per hour the train runs. $\therefore aN/1760N=$the fraction of an hour during which the poles are counted.

$\therefore 60aN/1760N=3a/88=$number of *minutes* during which the poles are counted.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; and W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let $x=$the number of minutes, and let $r=$number of miles per hour the train is running. Also, $1760/a=$number of poles in a mile, and $rx/60=$number of miles the train runs in x minutes. Then, $rx/60\times1760/a=88rx/3a=$number of poles passed in x minutes, or while the train is running $rx/60$ miles.

$\therefore 88rx/3a=r$; whence $x=3a/88$.

The number of minutes depends upon the distance the poles are apart irrespective of the rate of the train.

Also solved by *O. W. ANTHONY*, *P. S. BERG*, *A. H. HOLMES*, *C. D. SCHMITT*, *H. C. WILKES*, *B. F. YANNEY*, and *G. B. M. ZERR*.

61. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Demonstrate the identity $2^{2n+1}\frac{d^n}{dx^n}\left(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\right)=e^{\sqrt{x}}$.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

It may be proved inductively that $\frac{d^n}{dx^n}e^{\sqrt{x}}=\frac{1}{4x}\frac{d^{n-2}}{dx^{n-2}}e^{\sqrt{x}}-(n-\tfrac{3}{2})\frac{d^{n-1}}{dx^{n-1}}e^{\sqrt{x}}$.

Change n to $n+3$; then

$$\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}=\frac{1}{4x}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}-(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}.$$

Clearing of fractions and transposing,

$$\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+4x\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}.$$

Multiply through by $x^{n+\frac{1}{2}}$, and we have

$$x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4[(n+\tfrac{3}{2})x^{n+\frac{1}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+x^{n+\frac{3}{2}}\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}]$$

$$=4\frac{d}{dx}[x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}].$$

Multiply through by 2^{2n+1} and differentiate n times, and we have

$$2^{2n+1}\frac{d^n}{dx^n}(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}})=2^{2n+3}\frac{d^{n+1}}{dx^{n+1}}(x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}).$$

Hence, if the form given is true for n, it will be true for $n+1$. It may be easily verified that it is true for $n=2$. Therefore it is *generally* true.

II. Solution by HENRY HEATON, M. S., Atlantic, Iowa, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

$$\frac{d}{dx}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}},\ \frac{d^2}{dx^2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x^{\frac{3}{2}}},\ \frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8x^{\frac{5}{2}}},$$

$$\frac{d^4}{dx^4}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15)}{16x^{\frac{7}{2}}}.\quad \therefore x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8}.$$

$$\frac{d}{dx}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=\tfrac{1}{16}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^5\frac{d^2}{dx^2}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

$$\text{Also } x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}=\tfrac{1}{16}e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15).$$

$$\frac{d}{dx}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{32}e^{\sqrt{x}}(x-3\sqrt{x}+3),$$

$$\frac{d^2}{dx^2}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{64}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^7\frac{d^3}{dx^3}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

Hence generally $2^{2n+1}\frac{d^n}{dx^n}\{x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\}=e^{\sqrt{x}}$.

Also solved by *B. F. YANNEY*.

PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Given $\sqrt[3]{a+x}+\sqrt[3]{a-x}=\sqrt[3]{c}$ to find x.

71. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

When $x=0$, find the the limit of the expression

$$U=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

Prove geometrically:

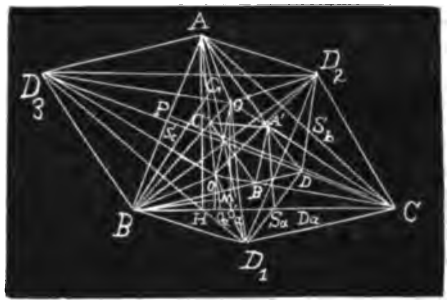
If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aA_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Since AC, DD_2 and BC, DD_1 bisect each other the quadrilaterals $ADCD_2$ and $BDCD_1$ are parallelograms, and AD_2, BD_1 both being equal and parallel to DC are equal and parallel to each other. Hence ABD_1D_2 is a parallelogram and AB is equal and parallel to D_1D_2. Similarly, AC is equal and parallel to D_1D_3, and BC is equal and parallel to D_2D_3.

$\therefore$ Triangle ABC is equal to triangle $D_1D_2D_3$. Also AD_1, BD_2, and

CD_3 intersect at the same point. For BD_2 and CD_3 bisect each other, also BD^2 and AD_1 bisect each other.



$\therefore$ BD_2, AD_1, and CD_3 bisect one another in the same point. Since triangle BDC=triangle D_3AD_2, DD_a=the perpendiculer distance from A to D_2D_3.

Draw AH, OO_a, $O'O'_a$, DD_a perpendicular to BC; then the point of intersection of the three lines AD_1, BD_2, CD_3 is distant from BC, $\frac{1}{2}(AH-DD_a)$.

$$DD_a=\frac{2b^2c^2.\triangle}{a(a^2b^2+a^2c^2+b^2c^2)}.\quad \text{(Schwatt's Curves, p. 10).}$$

$$AH.a=2\triangle.\quad \therefore AH=\frac{2\triangle}{a}.$$

$$\frac{AH-DD_a}{2}=\frac{\triangle.a(b^2+c^2)}{(a^2b^2+a^2c^2+b^2c^2)}=\frac{OO_a+O'O'_a}{2}.\quad \text{(Schwatt's Curves, p. 9).}$$

$\therefore$ AD_1, BD_2, CD_3 intersect at the mid-point of OO'.

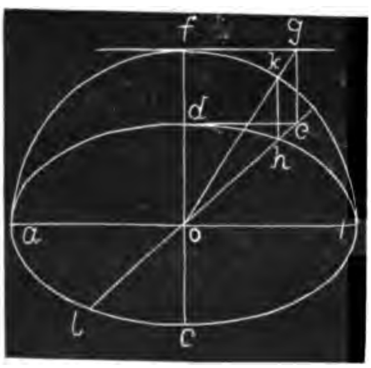
$\therefore$ Since AD_1, OO'; BD_2, OO'; CD_3, OO' all bisect one another, the quadrilaterals AOD_1O', BOD_2O', COD_3O' are parallelograms.

55. Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter lh forms with the major axis; it is required to find the length of this diameter.

I. Solution by the PROPOSER.

SOLUTION. Draw the semicircle afb on the diameter ab. Produce cd to f, and draw the tangents to the ellipse and the circle parallel to ab at the points d and f respectively. Produce lh to intersect de at e. Draw eg perpendicular to ab intersecting fg at g. Draw go intersecting the semicircle at k. Draw kh perpendicular to ab intersecting oe at h one extremity of the diameter lh.



ANALYSIS. The semiellipse adb may be considered as the projection on the plane of the paper of the semicircle afb, the latter being revolved about the diameter ab into a position when f is projected at d. The tangent fg which is parallel to ab is projected at de also parallel to ab. The points e and h are respectively the projections of g and k. Since the projection of every point on the

semicircle is found in a line drawn through it perpendicular to ab, the axis about which the semicircle revolves, kh drawn perpendicular to ab intersects oe at h and gives one point of the ellipse ; and therefore one extremity of the diameter lh.

II. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania; W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana; and J. O. MAHONEY, B. E., Assistant in Mathematics, Vanderbuilt University, Nashville, Tennessee.

If we denote ab by a, cd by b and tangent α by m, we have the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } y=mx \text{ which intersect at}$$

$$\left(\frac{ab}{\sqrt{b^2+a^2m^2}},\ \frac{abm}{\sqrt{b^2+a^2m^2}}\right) \text{ and } \left(-\frac{ab}{\sqrt{b^2+a^2m^2}},\ -\frac{abm}{\sqrt{b^2+a^2m^2}}\right),$$

the distance between these points being equal to

$$2ab\sqrt{\frac{1+m^2}{b^2+a^2m^2}}.$$

Also solved by *G. B. M. ZERR*, *J. SCHEFFER*, and *WILLIAM HOOVER*.

PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Onio University, Athens, Ohio.

Prove that the loci of the foci of the variable ellipses form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having for diameters the semi-latasrectum of the fixed ellipse.

61. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O and meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the three escribed circles.

62. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median of one of them* equal, each to each.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscate, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscate whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Solving the equatians, $r\cos\theta+a\cos2\theta=0$ (strophoid), and $r=e^2\cos2\theta$ (lemniscate), we find they coincide when $\sin\theta=\sqrt{\tfrac{1}{2}}$ (1), or $\sin\theta=\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}$ (2).

(1) shows that they coincide at the origin for all values of a and e. We have to find the relation between the axes a and e which will make the curves tangent at the points determined by (2), provided those points are on both the leaves. Let $\phi=\angle$ made by the tangent at any point, with the radius vector drawn to that point. Then by the formula $\tan\phi=r\dfrac{d\theta}{dr}$.

Now for the lemniscate $r=\pm e\sqrt{\cos2\theta}$. $\dfrac{dr}{d\theta}=\dfrac{\mp e\sin2\theta}{\sqrt{\cos2\theta}}$.

$$\tan\phi=\pm e\sqrt{\cos2\theta}\left(\frac{\sqrt{\cos2\theta}}{\mp e\sin2\theta}\right)=\frac{2\sin^2\theta-1}{2\sin\theta\sqrt{1-\sin^2\theta}}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For the strophoid $r=-a\cos2\theta/\cos\theta$.

$$dr/d\theta=-a(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)/\cos^2\theta.$$

$$\tan\phi=[a\cos^2\theta\cos2\theta]/[a\cos\theta(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)]$$

$$=[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]\ \ldots\ldots\ldots\ldots(4).$$

Now equate (3) and (4) and substitute from (1),

$$[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]=(2\sin^2\theta-1)/2\sin\theta\sqrt{1-\sin^2\theta}.$$

$$2\sin\theta(1-\sin^2\theta)(1-2\sin^2\theta)=\sin\theta(1-2\sin^2\theta)(3-2\sin^2\theta),$$

$$2\sqrt{\tfrac{1}{2}}(1-\tfrac{1}{2})(1-1)=\sqrt{\tfrac{1}{2}}(1-1)(3-1)\text{ or }0=0,$$

which shows that the curves are tangent at point ($\theta=\sin^{-1}\sqrt{\frac{1}{2}}$, $r=0$) for any value of a and e. Again substituting from (2),

$$2\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{a^2}{2a^2-e^2}\right)\left(\frac{e^2}{2a^2-e^2}\right)=\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{e^2}{2a^2-e^2}\right)\left(\frac{4a^2-e^2}{2a^2-e^2}\right).$$

This resolves into the three equations: $\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}=0$, whence $e=\pm a$..... (5);

$$\frac{e^2}{2a^2-e^2}=0, \text{ whence } e=0 \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

$$\frac{2a^2}{2a^2-e^2}=\frac{4a^2-e^2}{2a^2-e^2}, \text{ whence } e=\pm a\sqrt{2} \ldots\ldots\ldots\ldots (7).$$

From (5) substituted in (2), $\sin\theta=0$. $\therefore$ the curves are tangent at the extremity of the common axis, and the equations become,

$$r\cos\theta+a\cos2\theta=0 \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$r^2=a^2\cos2\theta \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (9) $r_1=\pm a\sqrt{\cos2\theta}$.

From (8) $r_2=\dfrac{-a\cos2\theta}{\cos\theta}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{\cos2\theta}{1-\sin^2\theta}}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$.

Since for any value of $\sin\theta$ numerically less than $\sqrt{\frac{1}{2}}$, $\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$ is numerically less than 1, r_2 is then numerically less than r_1. But by tracing the curves the leaf of each is seen to be formed by values of θ determined by this limit. $\therefore$ every point of the leaf of the strophoid lies within the lemniscate, and the former is in this case inscribed. From (6) equation of lemniscate becomes $r^2=0$, and the curve becomes a point. From (7) by substituting in (2) $\sin\theta=\sqrt{\dfrac{-a^2}{0}}$ an impossible value.

Accordingly the leaf of the strophoid can be inscribed in the leaf of the lemniscate when their axes are equal, and under no condition can the leaf of the lemniscate with an axis greater than 0 be inscribed in the leaf of the strophoid.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

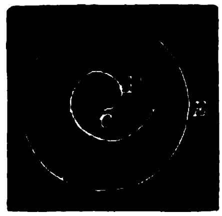
[It will be seen that Professor Black's result does not realise the intention of the problem as given by the Proposer. However, even for the Proposer's reading of the problem, his solution seems to us to be defective in several points. We may give Professor Zerr's solution later. EDITOR.]

59. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A draw bridge, a feet in length, moves uniformly about a center axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its center, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a=320$, and $n=2$.

Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College, Houston, Mississippi.

Let the man start at C and walk toward E, the table turning positively. He will traverse R, while the table turns $\frac{n}{2}.2\pi$. As velocities are uniform, we have,



$CP : PCE :: R : \pi n$, or $\rho : \theta :: R : \pi n$,

whence $\rho = \frac{R\theta}{\pi n}$ is the equation of the curve.

As $dl=[(\rho d\theta)^2+d\rho^2]^{\frac{1}{2}}$ we have,

$$dl=\frac{R}{\pi.n}(1+\theta^2)^{\frac{1}{2}}\,d\theta \text{ for this curve, and}$$

$$2l=\frac{2R}{\pi.n}\int_o^{\theta'}[1+\theta^2]^{\frac{1}{2}}\,d\theta, \text{ and}$$

$$L=\frac{2R}{\pi.n}\left[\frac{\theta}{2}(1+\theta^2)^{\frac{1}{2}}+\tfrac{1}{2}\log(\theta+\sqrt{1+\theta^2})\right]_o^{\pi.n}$$

$$L=\frac{R}{\pi.n}[\pi.n(1+\pi^2.n^2)^{\frac{1}{2}}+\log(\pi.n+\sqrt{1+\pi^2.n^2})].$$

If $a=100$ feet, and $n=2$,

$$L=\frac{50}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=338.3.$$

[No. 41, *Calculus.*]

If $a=36$ inches, and $n=1$, we have,

$$L=\frac{18}{\pi}[\pi\sqrt{1+\pi^2}+\log(\pi+\sqrt{1+\pi^2}]=69.6+ \text{ inches.}$$

[No. 45, *Calculus.*]

If $a=320$, and $n=2$, we have,

$$L=\frac{160}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=1082.56 \text{ feet.}$$

[No. 50, *Calculus.*]

The ratio of rates of extremity of the bridge and the man in his path is:

$$\frac{a}{2}\,d\theta\div dl=\frac{\pi n}{\sqrt{1+\theta^2}}.$$

The ratio of rates of extremity of bridge and the man's *walking* is:

$$\frac{\pi an}{a}=\pi n.$$

Also solved by *G. B. M. ZERR* and *C. W. M. BLACK*.

PROBLEMS.

57. Proposed by F. M. McGAW. A. M., Mathematical Department, Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation : $(1+x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with a velocity of V and the wheel rotates with velocity ω, what is the component of force or pressure along the axis if it is turned at an angle α to the direction of the wind, assuming the resistance of the wheel in turning to be R?

No solution of this problem has been received.

33. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

At what angle with the axis of a stalk must a *sharp* wedge-shaped blade be struck, in order to *sever* the stalk with the *least* force?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let φ be the inclination of the axis of the stalk to the blade. A=area of of section made by blade, r=radius of stalk, and suppose resistance per unit of area to vary as $f(\varphi)$.

$\therefore$ R=resistance per unit of area$=mf(\varphi)$.

$\therefore$ $A=\pi r^2\operatorname{cosec}\varphi$.

$\therefore$ Work of cutting any section is $\pi r^2 mf(\varphi)\operatorname{cosec}\varphi$. This may be made a minimum when $f(\varphi)$ is known.

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Since the blade is *sharp* we may neglect the force required to cut through the fibres and only regard that required to produce longitudinal compression.

Call k the coefficient of longitudinal compression, θ the angle of the blade, and ϕ the angle which the lower surface of the blade makes with the horizontal. Then when the blade has just cut through the stalk the force upon each surface parallel to the axis of the stalk will be

$$dF=k[\tan(\phi+\theta)-\tan\phi]xydx.$$

Resolving these parallel to the surface of wedge and parallel to median line of wedge we have—

$$dF_1=k.\frac{\cos(\phi+\theta)+\cos\phi}{\sin\frac{\theta}{2}}(\tan(\phi+\theta)-\tan\phi)xydx,$$

where F_1 is the force perpendicular to base of wedge.

$$\text{Then } F_1=k.\frac{\cos(\phi+\theta)+\cos\phi)}{\sin\frac{\theta}{2}}[\tan(\phi+\theta)-\tan\phi]\int_0^{2a}xydx$$

$$=2k\cos\frac{\theta}{2}\Big[\sec(\phi+\theta)+\sec\phi\Big]\int_0^{2a}xydx.$$

$$\frac{dF_1}{d\phi}=0 \text{ for minimum.}$$

$\therefore$ $\sec(\phi+\theta)\tan(\phi+\theta)+\sec\phi\tan\phi=0$. By some obvious reductions

$$\sin(\phi+\frac{\theta}{2}).\{\cos^2(\phi+\frac{\theta}{2})+\sec^2\frac{\theta}{2}\}=0,$$

whence $\phi=-\frac{\theta}{2}$.

That is, the medial line is horizontal. The second factor gives imaginary results, except when $\theta=0$.

PROBLEMS.

39. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Supposing the initial inclination of the swing to the vertical is α and that the person always crouches when in the highest position and stands up when in the lowest, his center of gravity moving through a distance b measured from lower part of swing upward, find how much the arc is increased after n complete vibrations.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The sum of three positive integral *cubic* roots of an equation is a square. What is the equation?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics and Science in Mississippi Normal College, Houston, Mississippi.

Let a, b, and c be the roots of the equation.

We then have $a^3+b^3+c^3=\square$.

This condition is satisfied by the equation $v^4(v^2+8v^2+27v^2)=\square$, where

$a^3=v^6$, $b^3=8v^6$ and $c^3=27v^6$. Forming the equation from the roots, we have: $x^3-(a^3+b^3+c^3)x^2+(a^3b^3+a^3c^3+b^3c^3)x-a^3b^3c^3=0$.

Substituting values of a, b, c and reducing, we have:

$x^3-36v^6x^2+251v^{12}x-216v^{18}=0$, where "$v$" may be 1, 2, 3, etc., in succession.

II. Solution by A. H. HOLMES, Brunswick, Maine, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas.

Let a, b, c be the roots of the cubic equation.

$\therefore x^3-(a+b+c)x^2+(ab+ac+bc)x=abc$, is the equation.

Let $a=5m^2$, $b=3m^2$, $c=m^2$. $\therefore 5m^2+3m^2+m=9m^2$.

$\therefore x^3-9m^2x^2+23m^4x=15m^6$ . . . .(1).

Let $a=m^2+mn$, $b=n^2-mn$, $c=2mn$, $m>n$.

$\therefore m^2+mn+n^2-mn+2mn=(m+n)^2$.

$\therefore x^3-(m+n)^2x^2+(3m^3n+3mn^3)x=2m^4n^2-2m^2n^4$ . . . .(2).

(1) and (2) both satisfy the conditions.

41. Proposed by H. C. WILKES, Skull Run, West Virginia.

Given $\dfrac{50(a+b)}{ab}=\dfrac{81(c+d)}{cd}$ . . . . . .(1); $\dfrac{56(a+c)}{ac}=\dfrac{75(b+d)}{bd}$ . . . . . .(2);

$\dfrac{65(b+c)}{bc}=\dfrac{66(a+d)}{ad}$ . . . . . .(3), to find the least integral values of a, b, c, d.

I. Solution by the PROPOSER.

The sum of equations (1), (2) and (3), after clearing of fractions, can be reduced to $20d(ab+ac+bc)=111abc$ . . . . . .(4).

Eliminating from (1) and (4), $6d=9c$.

Eliminating from (2) and (4), $5d=9b$.

Eliminating from (3) and (4), $4d=9a$.

$\therefore$ The numbers are in the ratio $a4$, $b5$, $c6$, $d9$, which will be the least integers that will satisfy the equation. [See problem No. 36.]

II. Solution by A. H. BELL, Hillsboro, Illinois.

Arranging,	$50acd+50bcd=81abc+81abd$.	(1).
	$75acd+56bcd=-75abc+56abd$.	(2).
	$65acd-66bcd=66abc-65abd$.	(3).
(1)×3	$150acd+150bcd=243abc+243abd$.	(4).
(2)×2	$150acd-112bcd=-150abc+112abd$.	(5).
(4)−(5)	$262bcd=393abc+131abd$.	(6).
(2)×13	$975acd-728bcd=-975abc+728abd$.	(7).
(3)×15	$975acd-990bcd=990abc-975abd$.	(8).
(7)−(8)	$262bcd=-1965abc+1703abd$.	(9).

(9)−(6), and reducing $3c=2d$. $\therefore c=2$, and $d=3$ (10).

These values in (1) and (2), etc., $a=4$ and $b=5$ (11).

To obtain the relative values between the two sets of values (10) and (11), take $(6)\times 1703-(9)\times 131$, results in $9a=4d$. $\therefore$ $a=4$ and $d=9$, $b=5$ and $c=6$. These are prime to each other. $\therefore$ are the least values.

III. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Nashville, Tennessee.

The equations can be written: $50\left(\frac{1}{a}+\frac{1}{b}\right)=81\left(\frac{1}{c}+\frac{1}{d}\right)$,

$56\left(\frac{1}{c}+\frac{1}{a}\right)=75\left(\frac{1}{b}+\frac{1}{d}\right)$, $65\left(\frac{1}{b}+\frac{1}{c}\right)=66\left(\frac{1}{a}+\frac{1}{d}\right)$.

Let $1/a=x$, $1/b=y$, $1/c=z$, and $1/d=u$, and the equations become $50x+50y-81z-81u=0$; $56x-75y+56z-75u=0$; $66x-65y-65z+66u=0$.

Thus we have three equations with four unknown quantities.

By determinants $x:y:z:u::$

$$\begin{vmatrix} 50, & -81, & -81 \\ -75, & 56, & -75 \\ -65, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & -81, & -81 \\ 56, & 56, & -75 \\ 66, & -65, & 66 \end{vmatrix} : \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & -75 \\ 66, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & 56 \\ 66, & -65, & -65 \end{vmatrix}$$

Evaluating the determinants, we have,

$x:y:z:u::(131)^2 90:(131)^2 72:(131)^2 60:(131)^2 40$,

or $x:y:z:u::90:72:60:40$.

Hence $1/a:1/b:1/c:1/d::90:72:60:40$,

or $a:b::c:d::4:5:6:9$;

whence $a=4$, $b=5$, $c=6$, $d=9$ are the lowest values.

Also solved by *A. H. HOLMES*.

PROBLEMS.

47. Proposed by EDMUND FISH, Hillsboro Illinois.

A rectangular field, whose length and breadth in rods are in whole numders, is enclosed with a fence and subdivided by fences on both diagonals, the total length of the fences is 2204 rods; required the sides and area.

48. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion $2:3:9$, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2=$the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

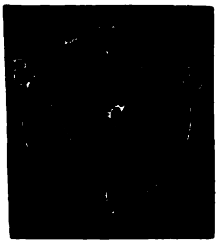
SOLUTIONS OF PROBLEMS.

30. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all the triangles which can be inscribed in a given circle.

I. Solution by the PROPOSER.

Let P_1OP_2 be any inscribed triangle; and through O draw any diameter OA. Two cases have now to be considered: (1), the triangle may lie *wholly* on one side of the diameter OA; (2), the triangle may lie *partly* on one side of the diameter OA.



I. Put $OA=2r$, $\angle AOP_1=\phi$, and $\angle AOP_2=\theta$; then $OP_1=2r\cos\phi$, $OP_2=2r\cos\theta$, and the area of the $\triangle$, P_1OP_2, $=A'$, $=2r^2\cos\phi\cos\theta\sin(\phi-\theta)$. Hence the average area of the triangles in this case, is

$$A_1=\int_0^{\frac{1}{2}\pi}\int_0^{\phi}A'd\phi d\theta\div\int_0^{\frac{1}{2}\pi}\int_0^{\phi}d\phi d\theta=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\phi\sin\phi\cos\phi d\phi$$

$$=\frac{r^2}{\pi^2}\Big[\sin2\phi-2\phi\cos2\phi\Big]_0^{\frac{1}{2}\pi}=\frac{r^2}{\pi}\text{............ (1).}$$

II. Put $\angle AOP_3=\psi$; then the area of the triangle P_2OP_3, $=A''$, $=2r^2\cos\theta\cos\psi\sin(\theta+\psi)$. Hence the average area of the triangles in this case, is

$$A_2=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}A''d\theta d\psi\div\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta d\psi=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta$$

$$+\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta\Big]=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\times\tfrac{1}{2}+\tfrac{1}{2}\times\frac{\pi}{4}\Big]=\frac{2r^2}{\pi}\text{............ (2).}$$

Hence the *required* average area becomes

$$A=\tfrac{1}{2}(A_1+A_2)=3r^2/2\pi\text{...... ....(3).}$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

We readily get the area of triangle

$$=\frac{R^2}{2}\{\sin2A+\sin2B+\sin2C\},$$

which, by virtue of the relation $A+B+C=\pi$, reduces to

$$\frac{R^2}{2}\{\sin 2A+\sin 2B-\sin 2(A+B)\}.$$

$$\therefore\ \text{Average area}=\frac{R^2}{2}\frac{\int_0^\pi\int_0^{\pi-A}\{\sin 2A+\sin 2B-\sin 2(A+B)\}dAdB}{\int_0^\pi\int_0^{\pi-A}dAdB}=\frac{3R^2}{2\pi}.$$

31. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Find the average length of a line drawn across the opposite sides of a rectangle, length l and breadth b.

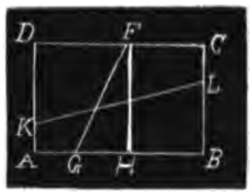
Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

Let $ABCD$ be the rectangle, FG the random line. Let $AB=l$, $BC=b$, $AH=x$, $AG=y$.

Then $FG=\{b^2+(x-y)^2\}^{\frac{1}{2}}$.

The limits of x are 0 and l; of y, 0 and x.

Hence the required average area is



$$\Delta=\frac{\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}dxdy}{\int_0^l\int_0^x dxdy}$$

$$=\frac{2}{l^2}\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}dxdy$$

$$=\frac{1}{l^2}\int_0^l\{x(b^2+x^2)^{\frac{1}{2}}+b^2\log[x+(b^2+x^2)^{\frac{1}{2}}]-b^2\log b\}dx$$

$$=\frac{1}{3l^2}(l^2+b^2)^{\frac{3}{2}}+\frac{b^2}{l}\log\{l+(l^2+b^2)^{\frac{1}{2}}\}-\frac{b^2}{l}\log b-\frac{1}{l^2}(l^2+b^2)^{\frac{1}{2}}-\frac{b^3}{3l^2}+\frac{b}{l^2}.$$

For the line KL, we get, by writing l for b and b for l,

$$\Delta_1=\frac{1}{3b^2}(l^2+b^2)^{\frac{3}{2}}+\frac{l^2}{b}\log\{b+(l^2+b^2)^{\frac{1}{2}}\}-\frac{l^2}{b}\log l-\frac{1}{b^2}(l^2+b^2)^{\frac{1}{2}}-\frac{l^3}{3b^2}+\frac{l}{b^2}.$$

Cor. I. If $l=b$, $\Delta=\frac{1}{3}(2l\sqrt{2})+l\log(1+\sqrt{2})-\frac{1}{l}\sqrt{2}-\frac{1}{3}l+\frac{1}{l}$.

Cor. II. If $l=b=1$, $\Delta=\frac{1}{3}(2-\sqrt{2})+\log(1+\sqrt{2})$, which is the same result as given in *Williamson's Integral Calculus*, page 409.

Also solved by *F. P. MATZ.*

PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

40. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn?

41. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A line is drawn at random across the chord and *given* arc of a circular segment. Find the mean area of the *divisions*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β=altitude, h, h_1 the hour angles.

$\therefore \sin\beta=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1$.

Also $h+\alpha=h_1+\alpha_1$.

But $\lambda=40^\circ$, $\alpha=343^\circ$, $\alpha_1=245^\circ\ 45'$, $\delta=-30^\circ\ 12'$, $\delta_1=-26^\circ\ 12'$.

$\therefore 66207\cos h-68734\cos h_1=3954\ldots\ldots\ldots\ldots(1)$.

$h_1-h=\alpha-\alpha_1=97^\circ\ 15'$, $\cos(h_1-h)=.12620\ldots\ldots\ldots\ldots(2)$.

Let $\cos h=x$, $\cos h_1=y$.

$\therefore$ from (2) $y=.1262x\pm\sqrt{.98407-.98407x^2}$. This in (1) gives $57532.7692x\mp68184.81534\sqrt{1-x^2}=3954$.

$\therefore x^2-.05716x=.58216$, $\therefore x=.79211$ or $-.73495$.

$\therefore h=37^\circ\ 37'$ or $137^\circ\ 18'12''$. $h=2$ hours, 30 minutes, 28 seconds, or 9 hours, 9 minutes, 12.8 seconds.

$\therefore$ sidereal time $=1$ hour, 22 minutes, 28 seconds, or 8 hours, 1 minute, 12.8 seconds.

36. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania.

"What is the length of a chord cutting off one-fifth of the area of a circle whose diameter is 10 feet?"

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let the chord subtend an angle $=2\theta$, $a=$radius of circle. Then the length of the chord $=2a\sin\theta$.

$\therefore a^2(\theta-\sin\theta\cos\theta)=\frac{1}{5}\pi a^2$.

$\therefore \theta-\sin\theta\cos\theta=\frac{1}{5}\pi$, $\therefore \theta=60^\circ 32'$ nearly.

$\therefore$ chord $=2a\sin\theta=10\sin\theta=8.7064$ feet.

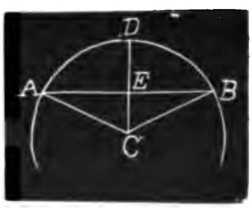
II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio, and Prof. P. S. BERG, Larimore, North Dakota.

Let $\theta=$the angle at the center, subtended by the required chord. Then $10\sin\theta=$the length of the required chord. Now $\frac{2\theta}{360}\pi 25$, the area of the sector, $-5\sin\theta\sqrt{(25-25\sin^2\theta)}$, the area of the triangle, $=5\pi$, the given area of the segment. Whence, by reduction, $\frac{\theta}{180}\pi-\sin\theta\cos\theta=\frac{\pi}{5}$.

$\therefore \frac{\theta}{90}\pi-2\sin\theta\cos\theta=\frac{2}{5}\pi$. $\therefore .0349065\theta-\sin 2\theta=1.256637$.

From which we readily find, by supposition, the value of θ; and from this, the value of $10\sin\theta$ to be 8.706, the length of the chord required.

III. Solution by A. H. BELL, Hillsboro, Illinois.



By Reversion of Series. Let the given diameter $=10=D$ and $1/5$ of circle $=a\pi r^2/d$, radius $=r$. To obtain the greatest convergency in the series, let ACB, the angle at the center $=2\theta$ and take the sector $ACD=r^2\theta/2$ and $r^2\sin\theta\cos\theta/2=ACE$.

Then $r^2(\theta-\sin\theta\cos\theta)/2=a\pi r^2/2d$ or arc$\theta=a\pi/d+\cos\theta\sqrt{(1-\cos^2\theta)}$ ....(1).

Make $\cos\theta=x$, and when expanded,

$$\theta=\frac{a\pi}{d}+x-\frac{x^3}{2}-\frac{x^5}{2.4}-\frac{3x^7}{2.4.6}-\frac{3.5x^9}{2.4.6.8}, \text{ etc.,}\ldots\ldots\ldots\ldots(2).$$

By trigonometry or calculus, we have,

$$\text{arc}\theta=\frac{\pi}{2}-x-\frac{x^3}{2.3}-\frac{3x^5}{2.4.5}-\frac{3.5x^7}{2.4.6.7}-\frac{3.5.7x^9}{2.4.6.8.9}, \text{ etc.,}\ldots\ldots(3).$$

(2)−(3) and ÷ by 2, etc.,

$$y=\frac{(d-2a)\pi}{4d}=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\text{ etc.,}\ldots\ldots\ldots(4).$$

Assume $x=Ay+By^3+Cy^5+Dy^7+Ey^9+$ etc.,……………………(5).

The powers of x substituted in (4); $y=Ay+$

$$\left(B-\frac{A^3}{6}\right)y^3+\left(C-\frac{A^2B}{2}-\frac{A^5}{40}\right)y^5+\left(D-\frac{A^2C}{2}-\frac{AB^2}{2}-\frac{A^4B}{8}-\frac{A^7}{112}\right)y^7+\text{etc.}$$

$\therefore A=1,\ B=1/6,\ C=13/120,\ D=493/5040,\ E=37369/362880$, etc., in (5).

$x=\cos\theta=y+y^3/6+13y^5/120+493y^7/5040+37369y^9/362880+$ etc.,,….(A).

Substituting values, $y=3\pi/20=0.471239=$logarithm $\bar{1}.673241+$.

2nd$=0.017441$
3rd$=0.002517$
4th$=0.000505$
5th$=0.000118$
Estimated$=0.000025$

$\cos\theta=0.491845$

2nd term $y^3=\bar{1}.019724-$
6 0.778151
$0.017441=\bar{2}.241573$

3rd term $y^5=\bar{2}.366206$
$13/120=\bar{1}.034762$
$0.002517+=\bar{3}.400968$

4th term $y^7=\bar{3}.712688$
$493/5040$….$\bar{2}.990416$
$0.000505-=\bar{4}.703104$

5th term $y^9=\bar{3}.059171$
$37369/362880$….$\bar{1}.012737$
$0.00018=\bar{4}.071908$

Chord $AB=10\sqrt{(1-\cos^2\theta)}=8.7068+$. $ACD=60^\circ 32' 17''$ nearly.

Note.—Formula (A) is also a general solution for the height of the circular segment (see problem 37, page 75, Vol. II). When the angle ACD is less than 50°, solve (1) for $\sin\theta$, and we have,

$$\theta<50^\circ=\sin\theta=\left(\frac{3a\pi}{2d}\right)^{\frac{1}{3}}-\tfrac{1}{10}\left(\frac{3a\pi}{2d}\right)^{\frac{3}{3}}-\tfrac{19}{1400}\left(\frac{3a\pi}{2d}\right)^{\frac{5}{3}}-\tfrac{71}{25200}\left(\frac{3a\pi}{2d}\right)^{\frac{7}{3}}-,\text{ etc.,}\ldots(B).$$

Chord$=D.\sin\theta$. It will be noticed that the convergency, in part, depends on the smallness of the value of y.

PROBLEMS.

43. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

43. Proposed by H. C. WILKES, Skull Run, West Virginia.

To find, if possible, a right angled triangle, the bisectors of the acute angles of which, can be expressed by integral whole numbers.

44. Proposed by Prof. P. S. BERG, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

EDITORIALS.

The August-September number of the MONTHLY will be issued about the 20th of September.

The address of Editor Finkel, after July 1st, will be The University of Chicago, Chicago, Illinois.

This issue has been delayed on account of our engravers missending the plate for Mr. Miller's portrait.

The University of Pennsylvania has conferred the degree of Doctor of Philosophy on our valued contributor, H. C. Whitaker. We congratulate Prof. Whitaker on this merited recognition of his ability.

PERIODICALS.

Annual Recreation Number of the Outlook. The Outlook Publishing Co., 13 Astor Place, New York City.

The Outlook's seventh annual Recreation Number contains nearly a hundred pages and scores of illustrations. Nearly all of the special articles relate to outdoor life, sport, recreation, and vacation possibilities. Among the writers are Ian Maclaren, the Rev. Dr. Henry van Dyke, the Rev. Dr. Charles H. Parkhurst, Kirk Munroe, General A. W. Greely, Poultney Bigelow, and many others.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. Review of Reviews Co., New York City.

The June number of *The Review of Reviews* is, as usual, full of the history of the important events that are taking place in various parts of the world. Dr. Shaw, the editor, has given a close analysis of the political situation which is now being worked out at St. Louis.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The June number of *The Cosmopolitan* is keeping up its literary merit, but is each time improving in the artistic excellence which it embodies.

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APPLICATIONS OF SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Paris, France.

Lagrange seems to have been the first to give a clear statement and at least a partial proof* of the following fundamental

THEOREM I. *The number* (N) *of different formal values which are obtained by permuting the n elements of a given function in every possible manner is a divisor of* $n!$.†

About thirty years later Ruffini proved that N cannot have the values 3 or 4 when $n=5$, in his work, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto,*" Bologna, 1799. He thus proved also that N cannot be equal to every divisor of $n!$.

As it was known that the value of N is the quotient obtained by dividing $n!$ by the order of the largest substitution group which transforms the function into itself it became an important problem to determine all the possible orders of the substitution groups of n elements, especially since it was believed that this would throw light on the solution of the general equation of the n^{th} degree. This problem has been solved only for small values of n.

The given theorem of Lagrange indicates the most direct application of substitution groups and therefore naturally furnished the starting point for the early investigations in this subject. It may be readily proved in the following manner.‡

Let ψ, the given function, be unchanged only by the substitutions in the

*The proof given by Lagrange in his article, "*Reflexions sur la resolution algebrique des equations,*" Memoires de l'Academie de Berlin, 1770 and 1771, seems to have been generally considered as complete. Cf. Mathieu, Comptes Rendus, 46, page 1047. Burkhardt, on the contrary, seems to regard it as incomplete. Cf. Zeitschrift fur Mathematik, 1892, page 141.

†$n!=1.2.3......n$.

‡Cf. Netto's Theory of Substitutions (Cole's edition) §41.

first row of the following rectangle. (These form a group, for the product of any two leaves ψ unchanged and is therefore found in this row.)

$$\begin{array}{llll}
s_1=1 & s_2 & s_3 \ldots\ldots \ldots\ldots s_\alpha & \\
t_2 & s_2t_2 & s_3t_2 \ldots\ldots\ldots\ldots s_\alpha\, t_2 & \\
t_3 & s_2t_3 & s_3t_3 \ldots\ldots\ldots\ldots s_\alpha\, t_3 & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
t_m & s_2t_m & s_3t_m \ldots\ldots\ldots\ldots s_\alpha\, t_m & m=\frac{n!}{\alpha}.
\end{array}$$

ψ will assume the same formal value if any one of the substitutions of a given row is applied to it, for the first factor leaves it unchanged and the second factor is the same throughout the row. If we assume that t_β ($\beta=2, 3, \ldots\ldots m$) is not found in a preceding row the substitutions of the rectangle are all different, for from

$$t_{\beta_2}s_{\gamma_2}=t_{\beta_1}s_{\gamma_1} \qquad (\beta_2 \overline{\overline{>}} \beta_1 \overline{\overline{<}} m)$$

we would have $\qquad (\gamma_1, \gamma_2, \gamma_3 \overline{\overline{<}} \alpha)$

$$t_{\beta_2}=t_{\beta_1}s_{\gamma_3}.$$

This is impossible unless $\beta_1=\beta_2$ and $s_{\gamma_3}=1$. In this case $t_{\beta_2}s_{\gamma_2}$ and $t_{\beta_1}s_{\gamma_1}$ occupy the same place in the given rectangle.

Since there are just $n!$ substitutions of n elements the given rectangle contains each substitution once and only once. If t_{β_1} and t_{β_2} would transform ψ into the same function (ψ_1) then would the products of all the substitutions in the rows containing t_{β_1} and t_{β_2} into a substitution* t_γ which transforms ψ_1 into ψ give 2α different substitutions that transform ψ into itself. This is contrary to the hypothesis. Therefore $N=m=$ a divisor of $n!$.

One of the best known functions to which these elementary principles of substitution groups are commonly applied is the anharmonic ratio of four points.† If the four points are represented by A, B, C, and D, their anharmonic or cross ratio may be represented by

$$\psi\equiv\frac{AB}{CB}\div\frac{AD}{CD} \text{ or } \frac{AB.CD}{AD.CB}.$$

It is required to find the number of formal values of ψ when the points are interchanged in every possible manner. We may do this by dividing $4!=24$ by the order of the largest group of degree four that transforms ψ into itself. Since ψ is unchanged by the substitution $AB.CD$ and also by the substitution

*Since the rectangle contains all the possible substitutions of n elements, it must contain the inverse of each of its substitutions. We shall always consider n to be a finite number.

†Cf. Harkness and Morley's.

$AD.BC$, it must be unchanged by the group generated by these substitutions, viz., by $(AB.CD)_4$. We know that there are only three groups* of degree four which include $(AB.CD)_4$ and that these contain either a substitution of the form AB or one of the form ABC. As no such substitution transforms ψ into itself $(AB.CD)_4$ is the largest group that has this property. The number of different values of ψ is therefore $24\div4=6$.

To find these six values of ψ we may arrange the substitutions of four elements as follows:

1	$AB.CD$	$AC.BD$	$AD.BC$
AB	CD	$ACBD$	$ADBC$
AC	$ABCD$	BD	$ADCB$
AD	$ABDC$	$ACDB$	BC
ABC	ACD	BDC	ADB
ACB	BCD	ABD	ADC

Since all the substitutions of a row transform ψ into the same function we can find the six formal values of ψ by applying to it the six substitutions of the first column in this rectangle.† We thus obtain the following, in order:

$$\frac{AB.CD}{AD.CB}=k;\ \frac{BA.CD}{BD.CA}=\frac{\dfrac{BA.CD}{AD.CB}}{\dfrac{AD.CB-AB.CD}{AD.CB}}=\frac{k}{k-1};\ \frac{CB.AD}{CD.AB}=\frac{1}{k}.$$

$$\frac{DB.CA}{DA.CB}=\frac{AD.CB-AB.CD}{AD.CB}=1-k;\ \frac{BC.AD}{BD.AC}=\frac{1}{1-k};\ \frac{CA.BD}{CD.BA}=\frac{k-1}{k}=1-\frac{1}{k}.$$

This example furnishes also a clear illustration of what we mean by "*different formal values.*" The six given values of ψ are all different as to form but k may have such values that they are not all really different. E. g., if $k=-1$, they coincide in pairs. In this case the ratio is called *harmonic.* If $k=$an imaginary cube root of -1, they coincide in triplets and the ratio is called *equianharmonic.*

It should be observed that each one of the four subgroups of $(ABCD)$all, which are of the form (ABC)all, has one substitution in each row. Hence the following

THEOREM II. *The six different formal values of an anharmonic ratio of four points may be obtained by transforming any three of its points symmetrically.*

---

*It is evident that the function is not symmetric. It would therefore only be necessary to examine it with respect to the other two groups.

†This is clearly only one of the 4096 different ways in which the six transforming substitutions may be selected.

If G is the largest group which transforms a function (ψ) into itself we say that *ψ belongs to G.* The same relation is also expressed by saying that *G belongs to ψ.** The former of these two expressions is to be preferred since only *one group belongs to any given ψ while an infinite number of functions belong to any given G.* This may be readily proved as follows :†

We first suppose that G is the symmetric group of n elements. Every symmetric function of these elements will then belong to G. That their number is infinite, follows directly from the fact that both a and b can have an infinite number of different values without impairing the symmetry of the following functions :‡

$$x_1^a+x_2^a+x_4^a+\ldots\ldots+x_n^a+bx_1x_2x_3\ldots\ldots x_n \;\ldots\ldots\ldots\ldots\; A.$$

We may now suppose that G consists of a single substitution, viz., identity. In this case every function of the n elements which is changed in form by each substitution of these elements belongs to G. If we suppose that a_1, a_2, a_3, ..., a_n represent n different given numbers, the following function belongs to G:

$$a_1x_1+a_2x_2+a_3x_3+\ldots\ldots+a_nx_n \;\ldots\ldots\ldots\ldots\ldots\ldots\; B.$$

We may now assign all the possible values of a_1 with the exception of the finite number of values represented by a_2, a_3, ..., a_n. In this way we obtain an infinite number of functions belonging to G.

We finally suppose that G represents any other group whose order is g. If we apply the substitutions of G to any one of the functions of B we obtain g different functions, ψ_1, ψ_2, ψ_3, ..., ψ_g. In any of the functions A we may suppose $n=g$ and the x's, in order, replaced by these ψ's. The resulting function belongs to this G. It is clear that we obtain an infinite number of such functions even by using a particular function of either A or B. We did not prove that all the functions belonging to G can be obtained in this way. In fact, this is not the case. As it follows from the definition that only one group belongs to a given function the proof is complete.

We have thus far only considered the relations between groups and functions when all the elements of the function which are permuted and no others are explicitly contained in the corresponding group. We have also only considered the number of values of a function when its elements are permuted according to the symmetric group. That the arguments which were employed apply to much more general cases may be illustrated by means of the following well-known Trigonometry formula

$$\text{Sin}\,\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

---

*Cf. Netto, Theory of Substitutions (Cole's Edition) §28.

†Ibid, §30.

‡ If a and b are complex numbers, A represents ∞^4 different functions.

If we regard the first member of this equation as a function of the three angles A, B, C of a triangle it belongs to the group (BC) and is therefore a three-valued function of the angles. The second member belongs to the group (bc) and is therefore a three-valued function of the sides. Hence the formula says that *a given three-valued function of the angles is equal to a given three-valued function of the sides.*

As no special properties were imposed upon any the sides or angles in deriving the formula the three different values of the angles must correspond separately to the three different values of the sides. It remains only to find the substitutions which transform the given formula into the other two. To do this we may arrange the substitutions of the angles and the sides, in the usual manner, as follows :

1	BC	1	bc
AB	ABC	ab	abc
AC	ACB	ac	acb

Since the substitutions of a row transform the corresponding functions in the same way and the rows of the two rectangles evidently correspond in order, we may effect the required transformation by any two substitutions such that one belongs to the first and the other to the second of the following two rows :

$$AB.ab,\ AB.abc,\ ABC.ab,\ ABC.abc$$

$$AC.ac,\ AC.acb,\ ACB.ac,\ ACB.acb$$

If we use the last one of each of these rows we have the rule frequently given in the text-books, viz., "The corresponding formulas for the other two angles may be obtained from this by the cyclical interchange of the letters."

The given formula might also be studied by employing a single group in place of two. The most convenient group is the intransitive group of degree six and order 36 which is obtained by multiplying the symmetric group of the angles into the symmetric group of the sides.* Since the given formula is transformed into itself only by the following four substitutions of this group

$$1,\ BC,\ bc,\ BC.bc,$$

it is a nine-valued function with respect to this group.† Since substitutions of this group transform the first member of the given formula into its three values without affecting the second member, these nine values may be arranged into three triplets, each of which has the same second member. By very simple trials we can show that six of these relations are absurd. Since three must be true the remaining relations are the required formulas.

*Cf. This journal, Vol. II, page 307.

†This may be proved in exactly the same way as Lagrange's theorem was proved.

Since similar remarks apply to a large number of the other Trigonometry formulas, it is clear that these formulas can be discussed in a more general and more definite manner by presupposing a thorough knowledge of the given elementary principles of substitution groups.

It is also easy to show that many problems of factoring can be discussed more completely by presupposing a knowledge of these groups. The following is a very simple illustration :

$$a^2-b^2-c^2+2bc=(a+b-c)(a-b+c) \ldots\ldots\ldots\ldots \ldots C.$$

The expression belongs to the group (bc) and is therefore a three-valued function; its factors belong to the groups (ab) and (ac) respectively and are therefore also three-valued functions. Hence C indicates an equality between a given three-valued function and the product of two other three-valued functions. These functions belong to three distinct groups. Arranging the substitutions of these groups in the usual manner, we have

1	*bc*	1	*ab*	1	*ac*
ab	*abc*	*ac*	*abc*	*ab*	*acb*
ac	*acb*	*bc*	*acb*	*bc*	*abc*

The three values of the given expression* may be obtained by applying to it one substitution from each of the three rows of the first rectangle, e. g., the first column. The factors of these transforms may evidently be found by applying the same substitutions to the given factors. Since ab and ac transform one of the factors into itself it follows that the three conjugate expressions contain only three distinct linear factors, viz., the three values of any one of them.

These observations indicate how we may readily determine the total number of substitutions by means of which the factors of all the conjugates of a given expression may be found from those of the given expression. They have brought us in contact with, at least, three important questions, viz. :

1. What relations exist between the factors of a system of conjugate expression ?

2. What relations exist between the groups of the factors and the group of the expression ?

3. To what extent may these relations be utilized in the process of factoring ?

*The same idea is expressed by "the three conjugates of the given expression" or by "the three transforms of the given expression."

THE BINOMIAL THEOREM.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Texas.

I use the following rule for expanding all binomials, whether the exponent is integral or fractional, positive or negative.

The number of terms of a binomial expansion is one more than the exponent when the exponent is a positive integer, otherwise the number of terms is infinite. For the first term of the expansion, raise the first term of the binomial to the required power. For any other term of the expansion, multiply the preceding term by the second term of the binomial, and this product by the exponent of the power diminished by two less than the number of terms from the beginning, divide this product by the product of the first term of the binomial into one less than the number of terms from the beginning, always observing the proper algebraic signs of the binomial terms.

$$(ax+by)^m=(ax)^m+\frac{m(ax)^m by}{ax}+\frac{m(m-1)(ax)^m(by)^2}{1.2.(ax)^2}+\ldots\ldots$$

$$+\ldots\ldots+\frac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}+\ldots\ldots(A).$$

(A) gives the expansion without reducing the terms.

(1). To expand $(3x\pm4y)^5$.

1st term $=(3x)^5=243x^5$; 2nd term $=\dfrac{243x^5\times(\pm4y)\times5}{3x}=\pm1620x^4y$;

$$\text{3rd term}=\frac{\pm1620x^4y\times(\pm4y)\times4}{2.3x}=4320x^3y^2\ ;$$

$$\text{4th term}=\frac{4320x^3y^2\times(\pm4y)\times3}{3.3x}=\pm5760x^2y^3\ ;$$

$$\text{5th term}=\frac{\pm5760x^2y^3\times(\pm4y)\times2}{4.3x}=3840xy^4\ ;$$

$$\text{6th term}=\frac{3840xy^4\times(\pm4y)\times1}{5.3x}=\pm1024y^5.$$

$$\therefore\ (3x\pm4y)^5=243x^5\pm1620x^4y+4320x^3y^2\pm5760x^2y^3+3840xy^4\pm1024y^5.$$

(2). To expand $(a^2+2b)^7$.

$$1\text{st term}=(a^2)^7=a^{14};\ 2\text{nd term}=\frac{a^{14}.2b.7}{a^2}=14a^{12}b;$$

$$3\text{rd term}=\frac{14a^{12}b.2b.6}{2.a^2}=84a^{10}b^2;\ 4\text{th term}=\frac{84a^{10}b^2.2b.5}{3.a^2}=280a^8b^3;$$

$$5\text{th term}=\frac{280a^8b^3.2b.4}{4.a^2}=560a^6b^4;\ 6\text{th term}=\frac{560a^6b^4.2b.3}{5.a^2}=672a^4b^5;$$

$$7\text{th term}=\frac{672a^4b^5.2b.2}{6.a^2}=448a^2b^6;\ 8\text{th term}=\frac{448a^2b^6.2b.1}{7.a^2}=128b^7.$$

$$\therefore\ (a^2+2b)^7=a^{14}+14a^{12}b+84a^{10}b^2+280a^8b^3$$
$$+560a^6b^4+672a^4b^5+448a^2b^6+128b^7.$$

(3). To expand $(2+x)^{-3}$.

$$1\text{st term}=(2)^{-3}=\tfrac{1}{8};\ 2\text{nd term}=\tfrac{1}{8}\times\frac{x\times(-3)}{2}=-\frac{3x}{16};$$

$$3\text{rd term}=-\frac{3x}{16}\times\frac{x\times(-4)}{2.2}=\frac{3x^2}{16};\ 4\text{th term}=\frac{3x^2}{16}\times\frac{x\times(-5)}{3.2}=-\frac{5x^3}{32};$$

$$5\text{th term}=-\frac{5x^3}{32}\times\frac{x\times(-6)}{4.2}=\frac{15x^4}{128}.$$

$$\therefore\ (2+x)^{-3}=\tfrac{1}{8}-\frac{3x}{16}+\frac{3x^2}{16}-\frac{5x^3}{32}+\frac{15x^4}{128}-\ldots\ldots$$

(4). To expand $(1+\frac{2x}{3})^{\frac{3}{2}}$.

$$1\text{st term}=(1)^{\frac{3}{2}}=1;\ 2\text{nd term}=\frac{1.\frac{2x}{3}.\frac{3}{2}}{1}=x;$$

$$3\text{rd term}=\frac{x.\frac{2x}{3}.\frac{1}{2}}{2.1}=\tfrac{1}{6}x^2;\ 4\text{th term}=\frac{\frac{1}{6}x^2.\frac{2x}{3}.(-\frac{1}{2})}{3.1}=-\tfrac{1}{54}x^3;$$

$$5\text{th term}=\frac{-\frac{1}{54}x^3.\frac{2x}{3}.(-\frac{3}{2})}{4.1}=\tfrac{1}{216}x^4.$$

$\therefore\ (1+\frac{2x}{3})^{\frac{3}{2}} = 1+x+\frac{1}{6}x^2-\frac{1}{54}x^3+\frac{1}{216}x^4- \ \ldots\ldots$

(5). To expand $(8+12a)^{\frac{2}{3}}$.

$$\text{1st term} = (8)^{\frac{2}{3}} = 4;\ \text{2nd term} = \frac{4.12a.\frac{2}{3}}{8} = 4a\ ;$$

$$\text{3rd term} = \frac{4a.12a.(-\frac{1}{3})}{2.8} = -a^2;\ \text{4th term} = \frac{-a^2.12a.(-\frac{4}{3})}{3.8} = \frac{2a^3}{3};$$

$$\text{5th term} = \frac{\frac{2a^3}{3}.12a.(-\frac{7}{3})}{4.8} = -\frac{7a^4}{12}.$$

$\therefore\ (8+12a)^{\frac{2}{3}} = 4+4a-a^2+\frac{2}{3}a^3-\frac{7}{12}a^4+\ldots\ldots$

(6). To expand $(4a-8x)^{-\frac{1}{2}}$.

$$\text{1st term} = (4a)^{-\frac{1}{2}} = \frac{1}{2a^{\frac{1}{2}}};\ \text{2nd term} = \frac{1}{2a^{\frac{1}{2}}}.\frac{(-8x)(-\frac{1}{2})}{4a} = \frac{x}{2a^{\frac{3}{2}}};$$

$$\text{3rd term} = \frac{x}{2a^{\frac{3}{2}}}.\frac{(-8x)(-\frac{3}{2})}{2.4a} = \frac{3x^2}{4a^{\frac{5}{2}}};\ \text{4th term} = \frac{3x^2}{4a^{\frac{5}{2}}}.\frac{(-8x)(-\frac{5}{2})}{3.4a} = \frac{5x^3}{4a^{\frac{7}{2}}};$$

$$\text{5th term} = \frac{5x^3}{4a^{\frac{7}{2}}}.\frac{(-8x)(-\frac{7}{2})}{4.4a} = \frac{35x^4}{16a^{\frac{9}{2}}}.$$

$$\therefore\ (4a-8x)^{-\frac{1}{2}} = \frac{1}{2a^{\frac{1}{2}}}+\frac{x}{2a^{\frac{3}{2}}}+\frac{3x^2}{4a^{\frac{5}{2}}}+\frac{5x^3}{4a^{\frac{7}{2}}}+\frac{35x^4}{16a^{\frac{9}{2}}}+\ldots\ldots$$

$$= \frac{1}{2a^{\frac{1}{2}}}\left(1+\frac{x}{a}+\frac{3x^2}{2a^2}+\frac{5x^3}{4a^3}+\frac{35x^4}{8a^4}+\ldots\ldots\right).$$

The r^{th} term in (A) is $\dfrac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}$.

(7). Find the 4th term of $(\frac{a}{3}+9b)^{10}$.

$$m=10,\ r=4.\quad \therefore\ \text{4th term} = \frac{10\times 9\times 8\times(\frac{a}{3})^{10}(9b)^3}{1.2.3.(\frac{a}{3})^3} = 40a^7b^3.$$

(8). Find the 28th term of $(5x+8y)^{30}$.
$m=30$, $r=28$.

$$\therefore\ 28\text{th term}=\frac{30.29.28\ldots\ldots 6.5.4.(5x^{30}(8y)^{27}}{1.2.3\ldots\ldots 26.27.(5x)^{27}}=\frac{\underline{|30}}{\underline{|27}\ \underline{|3}}(5x)^{3}(8y)^{27}.$$

(9). Find the 8th term of $(1+2x)^{-\frac{1}{2}}$.
$m=-\frac{1}{2}$, $r=8$.

$$\therefore\ 8\text{th term}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})(-\frac{11}{2})(-\frac{13}{2})(1)^{-\frac{1}{2}}(2x)^{7}}{1.2.3.4.5.6.7.(1)^{7}}=-\frac{429x^{7}}{16}.$$

(10). Find the 10th term of $(1+3a^{2})^{\frac{16}{3}}$.
$m=\frac{16}{3}$, $r=10$.

$$\therefore\ 10\text{th term}=\frac{\frac{16}{3}.\frac{13}{3}.\frac{10}{3}.\frac{7}{3}.\frac{4}{3}.\frac{1}{3}(-\frac{2}{3}).(-\frac{5}{3}).(-\frac{8}{3})(1)^{\frac{16}{3}}(3a^{2})^{9}}{1.2.3.4.5.6.7.8.9.(1)^{9}}=-\frac{1040a^{18}}{81}.$$

(11). Find the 5th term of $(3a-2b)^{-1}$.
$m=-1$, $r=5$.

$$\therefore\ 5\text{th term}=\frac{(-1)(-2)(-3)(-4)(3a)^{-1}(2b)^{4}}{1.2.3.4.(3a)^{4}}=\frac{16b^{4}}{243a^{5}}.$$

These are enough examples to illustrate both the rule and the general term.

I have used this method with my classes for several years and find it easier and better than any other method I have ever used. I have never seen this method in this form. If any of the readers of the MONTHLY have ever seen it, I would be pleased to know where to find it.

A GEOMETRICAL PROOF THAT $0\times\infty$ IS INDETERMINATE.

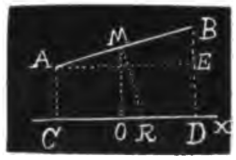
By B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

The Proof that I shall offer is not new perhaps, but I have never seen it in print, and for that reason I shall give it in the MONTHLY.

The proof follows as a corollary of the following

PROPOSITION : *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*

Let AB be the straight line revolved about CD as an axis. When AB is not parallel to the axis CX, the surface generated by AB is the convex surface of the frustum of a cone.



$\therefore$ Area generated by $AB=AB\times 2\pi MO$.

But $MO : AE :: MR : AB$, or

$$AB\times MO=AE\times MR=CD\times MR.$$

$\therefore$ Area generated by $AB=CD\times 2\pi MR$. Now if AB is made to approach perpendicularity, MR will approach parallelism to CX, and, in consequence, CD will approach 0 as its limit and MR will approach ∞ as its limit. Hence, in the limit, we have

$$\text{area generated by } AB=CD\times 2\pi MR=0\times 2\pi\times\infty.$$

But area generated by AB when AB is perpendicular to CX is $\pi(BC^2-AC^2)$. Hence, $\pi(BC^2-AC^2)=0\times 2\pi\times\infty$, or $BC^2-AC^2=2\times 0\times\infty=0\times\infty$. When $AC=0$, we have $0\times\infty=BC^2$. Now BC is entirely arbitrary. Hence, $0\times\infty$ is indeterminate. But when BC is a definite quantity, as for example 3, then $0\times\infty$ has the definite value 9.

The fundamental type of symbols of indetermination is $\frac{0}{0}$, and to this type $0\times\infty$ may be reduced. Thus, $0\times\infty=0\times\frac{1}{\frac{1}{\infty}}=\frac{0\times 1}{0}=\frac{0}{0}$. The indeterminate form, $\frac{\infty}{\infty}$, $=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$. Also $\infty-\infty=\frac{1}{\frac{1}{\infty}}-\frac{1}{\frac{1}{\infty}}=\frac{1}{0}-\frac{1}{0}=\frac{0}{0}$; $0^0=0^a\div 0^a=\frac{0^a}{0^a}=\frac{0}{0}$; $\infty^0=\infty^a\div\infty^a=\frac{\infty}{\infty}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$.

When these forms occur as the answers of problems, they have, in general, perfectly definite values, and these definite values must be found. But when these forms stand apart from the consideration of problems, they are perfectly meaningless,

Drury College, September 14, 1896.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A pipe 1 foot long and 27-32 inch in diametèr has a half-inch orifice and weighs $1\frac{3}{4}$ pounds. What is the diameter of a pipe the same length and orifice, but weighing 41 ounces?

I. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Let V_1 = volume of "solid" pipe.

Let V_2 = volume of bore.

Then $V_1 - V_2$ = volume of metal $= \pi(\frac{1419}{1024})$ cubic inches.

Since weights are proportional to volumes, $\pi(\frac{1419}{1024}) : V_3 = 28 : 41$, where V_3 = volume of required size of pipe.

Add to this volume of bore $= V_2$, and we have,

$$V_3 + V_2 = V_4 = \text{new "solid" pipe} = \pi(\tfrac{79683}{28672}) \text{ cubic inches.}$$

Hence $R = \{\pi(\frac{79683}{28672}) \div \pi.12\}^{\frac{1}{2}} = .431\frac{192}{896}$ and $D = .9625$ inches.

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, New Jersey.

The volumes of the two pipes will have the same ratio as their weights.

$$\text{Hence, } \frac{\pi l\left[\left(\frac{D}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]}{\pi l\left[\left(\frac{D'}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]} = \frac{w}{w'}; \text{ or } \frac{D^2 - d^2}{D'^2 - d^2} = \frac{w}{w'},$$

where the D's represent the diameters of the pipes, and d the common diameter of their *orifices*. From this

$$D'=\sqrt{\frac{w'D^2-w'd^2+wd^2}{w}}=\sqrt{\frac{w'}{w}(D^2-d^2)+d^2}.$$

$$=\sqrt{\tfrac{41}{28}[(\tfrac{27}{32})^2-(\tfrac{1}{2})^2]+\tfrac{1}{4}}=\sqrt{\tfrac{26561}{28672}}=.96248+\text{ inches}.$$

Also solved by *G. B. M. ZERR*, *H. C. WILKES*, and *J. SCHEFFER*.

61. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Insured my store for a/bth$=\frac{3}{4}$th of its value, at $r=1\frac{1}{4}\%$; but soon afterward the store was burned down, and my loss over the insurance was $\$L=\$4150$. What was the value of my store?

I. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Construing the terms of this question as they are used in legal and insurance circles the solution is $\$4,150\times4=\$16,600$.

But the proposer evidently intends to reckon the premium paid as a part of the "loss."

Then for every $4.00 of value $3.00 was insured at a cost of 3.75 cents, leaving $1.0375 of loss.

Hence 1.0375 : 4 :: 4150 : 16,000.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The value of the policy is $\frac{a}{b}\cdot\frac{r}{100}x$, x representing the value of the store.

We have, therefore, obviously,

$$x\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right]=L,\ \therefore x=L\div\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right].$$

Substituting numerical value, we find $x=\$16,000$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and A. P. READ, A. M., Clarence, Missouri.

Let s=value of store.

Then $s-\frac{as}{b}=\left(\frac{b-a}{b}\right)s$; $\frac{r}{100}\times\frac{as}{b}=\frac{ars}{100b}$. $\therefore\left(\frac{b-a}{b}+\frac{ar}{100b}\right)s=L$.

$$\therefore s=\frac{100bL}{100b-100a+ar}=\frac{400(4150)}{100+\frac{15}{4}}=\$16,000.$$

Also solved by *EDWARD R. ROBBINS*, and *F. M. McGAW*.

PROBLEMS.

65. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at \$5.37½ per yard, less 12½ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at 5½%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds m=10% of water to the pure wine he buys, and then sells the mixture at a price n=10% greater than the cost price of the pure wine. What is his rate per cent. of profit?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that every algebraic equation can be transformed into another equation of the same degree, but which wants its n^{th} term.

I. Solution by HENRY HEATON, M. Sc., County Surveyor, Atlantic, Iowa.

To illustrate, let $x^4+ax^3+bx^2+cx+d=0$ be any equation of the fourth degree. Put $x=y+p$; then the equation becomes

$$y^4+(4p+a)y^3+(6p^2+3ap+b)y^2+(4p^3+3ap^2$$
$$+2bp+c)y+p^4+ap^3+bp^2+cp+d=0.$$

Since we are at liberty to give p any value, we may give it the value that will make $4p+a=0$ or $-a/4$; then will the coefficient of y^3 disappear. It is also evident that we may give p such a value, that any desired coefficient will disappear. It is also evident that to find the desired value of p by this method requires for the second term, the solution of an equation of the first degree; for the third term, the solution of an equation of the second degree, etc. It is further evident that this is true without regard to the degree of the original equation.

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If not already so, any equation of the n^{th} degree may be reduced to the form $x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+L=0$. Now, by putting for x, $x+a$, we obtain a new equation whose roots differ from the corresponding roots of the given equation by a, (and whose degree, therefore, is still the n^{th}) viz.:

$$x^n+(na+A)x^{n-1}+\left(\frac{n(n-1)}{\underline{|2}}a^2+(n-1)Aa+B\right)x^{n-2}$$

$$+\ldots\ldots+(a^n+Aa^{n-1}+Ba^{n-2}+\ldots\ldots+L=0.$$

As a is an arbitrary constant, it may be selected so that $(na+A)=0$, or

$$\left(\frac{n(n-1)}{\underline{|2}}a^2+(n-1)Aa+B\right)=0,$$

or any coefficient, except the first, $=0$. Hence, any term, except the first, may thus be removed.

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Every algebraic equation may be written

$$X^k-\Sigma\alpha.X^{k-1}+\Sigma\alpha\beta.X^{k-2}\ldots\ldots=0.$$

The coefficient of the n^{th} term will be $\Sigma\alpha\beta\gamma\ldots\ldots$to $n-1$ factors. Now in place of X write $X+h$; then α, β, γ, etc., will be changed into $\alpha+h$, $\beta+h$, $\gamma+h$, etc. The coefficient of n^{th} term will then be $\pm\Sigma(\alpha+h)(\beta+h)(\gamma+h)\ldots\ldots$to $n-1$ terms. If we equate this to zero, we may consider it an equation of degree $n-1$ in h. This will give $n-1$ values of h. Therefore there are $n-1$ transformations which will make the n^{th} term vanish. Consider the first term, $n-1$; there are in that case no transformations.

Also solved by *PROF. E. W. MORRELL.*

63. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Given $x^2+x\sqrt{xy}=10$, and $y^2+y\sqrt{xy}=20$ to find x and y by quadratics.

I. Solution by E. L. BROWN, A. M., Professor of Mathematics, Capital University, Columbus, Ohio; HENRY HEATON, M. Sc., Atlantic, Iowa; and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Factoring, we have $x^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=10$, $y^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=20$.

$$\therefore y^{\frac{3}{2}}/x^{\frac{3}{2}}=2,\ y^{\frac{3}{2}}=2x^{\frac{3}{2}}.\quad \therefore y=\sqrt[3]{4}x.$$

$\therefore\ y^{\frac{1}{3}}=\pm x^{\frac{1}{3}}\sqrt[3]{2}$, this in either equation gives

$$x^2(1\pm\sqrt[3]{2})=10,\quad \therefore\ x=\pm\sqrt{\frac{10}{1\pm\sqrt[3]{2}}},\ y=\pm\sqrt[3]{4}\sqrt{\frac{10}{1\pm\sqrt[3]{2}}}.$$

II. **Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; and M. A. GRUBER, A. M., War Department, Washington, D. C.**

Factoring the given equations, we obtain

$x\sqrt{x}(\sqrt{x}+\sqrt{y})=10=a,$ ....(1), $\quad y\sqrt{y}(\sqrt{x}+\sqrt{y})=20=b,$ ....(2).

$$(1)\div(2)\text{ gives }\frac{x\sqrt{x}}{y\sqrt{y}}=\frac{a}{b}.\quad\text{Squaring and reducing, we get}$$

$$y=\frac{x\sqrt[3]{b^2}}{\sqrt[3]{a^2}},\text{ and }\sqrt{xy}=\frac{x\sqrt[3]{b}}{\sqrt[3]{a}}.$$

Substituting in first given equation, we have $x^2+\dfrac{x^2\sqrt[3]{b}}{\sqrt[3]{a}}=a$;

$$\text{whence }x=\pm\left(\frac{a\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}},$$

$$\text{and }y=\pm\left(\frac{b\sqrt[3]{b}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

III. **Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio; and Prof. E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont.**

The given equations may be written,

$x\sqrt{xy}=10-x^2$ ....(1), $\quad y\sqrt{xy}=20-y^2$ ....(2).

$(1)\times(2)$, $x^2y^2=200-20x^2-10y^2+x^2y^2$. $\quad\therefore\ 2x^2+y^2=20$ ....(3).

From (2) and (3), $2x^2=y\sqrt{xy}$. $\therefore\ y=x\sqrt[3]{4}$ ....(4).

$$(4)\text{ in }(1),\ x=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}}.\quad\therefore\ y=\pm\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

IV. **Solution by J. H. DRUMMOND, LL. D., Portland, Maine; A. H. HOLMES, Brunswick, Maine; and O. W. ANTHONY, M. Sc., New Windsor College, New Windsor, Maryland.**

Let $y=v^2x$, then $x^2(1+v)=a=10$, and $v^3x^2(1+v)=b=20$.

$$\therefore\ v=\sqrt[3]{\frac{b}{a}},\text{ and }x=\pm\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}},\ =\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

$$y=\pm\frac{b^{\frac{3}{4}}}{\sqrt{a^{\frac{1}{2}}+b^{\frac{1}{2}}}},\quad =\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

V. Solution by CHAS. A. HOBBS, A. M., Master of Mathematics in the Belmont School, Belmont, Massachusetts.

$$x^2+x^{\frac{4}{3}}y^{\frac{2}{3}}=10,\ y^2+x^{\frac{2}{3}}y^{\frac{4}{3}}=20.\quad \text{Let } y=vx.$$

$$\text{Then } x^2+v^{\frac{2}{3}}x^2=10,\ v^2x^2+v^{\frac{4}{3}}x^2=20.$$

$$\therefore x^2=\frac{10}{1+v^{\frac{2}{3}}},\text{ and } x^2=\frac{20}{v^2+v^{\frac{4}{3}}}.\quad \therefore \frac{10}{1+v^{\frac{2}{3}}}=\frac{20}{v^2+v^{\frac{4}{3}}}.$$

Dividing by 10, and clearing of fractions, $v^{\frac{4}{3}}=2$, $v=2^{\frac{3}{4}}$.

$$\therefore x^2=\frac{10}{1+2^{\frac{1}{2}}},\ x=\sqrt{\frac{10}{1+\sqrt[3]{2}}}.\quad y=2^{\frac{3}{4}}\sqrt{\frac{10}{1+\sqrt[3]{2}}}=\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

VI. Solution by J. W. WATSON, Middle Creek, Ohio; and H. C. WILKES, Skull Run, West Virginia.

Put $x=m^2$, $y=n^2$. Then, the given equations become, after factoring,

$m^3(m+n)=10$ ....(1), and $n^3(m+n)=20$........(2). Whence $n=m\sqrt[3]{2}$.

Then in (1) $m^3(m+m\sqrt[3]{2})=10$, or $m^4(1+\sqrt[3]{2})=10$.

$$\therefore m^4=\frac{10}{1+\sqrt[3]{2}},\text{ and } m^2=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}},=x.$$

Also, n^2, $=y$, $=\pm\sqrt{\dfrac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}$.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

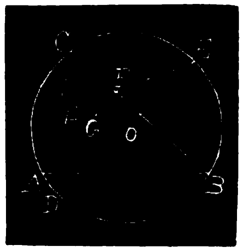
SOLUTIONS OF PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let O and H be the circum and ortho-centers respectively of the triangle ABC. Draw the diameter DE, connect E and H, and from F the mid-point of EH draw FG parallel to OE.



Now H and O are inverse points.

G is the mid-point of HO and $GF=\frac{1}{2}OE=$a constant.

$\therefore$ G is the center and GF the radius of the nine-point circle.

$\therefore$ The locus of F is the nine-point circle.

II. Solution by the PROPOSER.

Let $l\alpha+m\beta+n\gamma=0$..........(1) be any diameter. The isogonal transformation of (1) is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0 \qquad (2).$$

Now (1), passing through the center of the circumcircle, the coordinates of which are proportional to $\cos A$, $\cos B$, $\cos C$, gives the relation

$$l\cos A+m\cos B+n\cos C=0 \qquad (3).$$

Also, the center of (2), which is an equilateral hyperbola, with condition (3), is given by

$$\frac{l}{n}=\frac{-a\alpha^2+b\alpha\beta+c\alpha\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma},\quad \frac{m}{n}=\frac{a\alpha\beta-b\beta^2+c\beta\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma} \qquad (4).$$

Dividing (3) by n, and substituting equations (4), and reducing,

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta-a\alpha^2\cos A-b\beta^2\cos B-c\gamma^2\cos C=0 \qquad (5),$$

the nine-points circle.

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line, may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

If the four points be A, B, C, D, and the axis of x coincide with the given straight line, A, B may be supposed given by

$$\alpha x^2+2\beta x+\gamma=0 \ldots \ldots \ldots (1),$$

$$\text{or } x=\frac{-\beta\pm\sqrt{\beta^2-\gamma^2}}{\alpha} \ldots \ldots \ldots (2),$$

$$\text{and } C,\ D, \text{ by } \alpha' x^2+2\beta' x+\gamma'=0 \ldots \ldots \ldots (3).$$

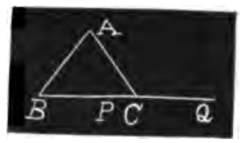
Now as long as γ exceeds β, (2) gives imaginary values for x, and so for a like pair of values for (3), which does not violate the condition

$$\alpha\gamma'+\alpha'\gamma=2\beta\beta' \ldots \ldots \ldots (4),$$

any number of values of β, γ in (2) always being consistent with (4).

II. **Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics, Indiana University, Bloomington, Indiana.**

Using trilinear coordinates, take B and C for the two real points on the real line $\alpha=0$, i. e., $b\beta+c\gamma=2\triangle$. $B^2+K^2\gamma^2=0$, is the equation of two lines through A'; that is $\beta+Ki\gamma=0$, and $\beta-Ki\gamma=0$. These lines form with $\beta=0$ (AC) and $\gamma=0$ (AB) a harmonic pencil, and hence intersect BC in two points forming with B and C a harmonic range.



Moreover these lines are imaginary for all real values of K and hence must intersect BC in imaginary points, otherwise they would contain two real points, which is impossible.

The coordinates of the points of intersections of these imaginary lines may be found by solving with $b\beta+c\gamma=2\triangle$. Thus $\beta=-Ki\gamma$ gives

$$(c-bKi)\gamma=2\triangle \text{ and } \gamma=\frac{2\triangle c}{c^2+b^2K^2}+\frac{2\triangle Kb}{c^2+b^2K^2}i$$

$$\text{and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}-\frac{2\triangle Kc}{c^2+b^2K^2}i, \text{ and } \beta=Ki\gamma, \text{ gives}$$

$$\gamma=\frac{2\triangle c}{c^2+b^2K^2}-\frac{2\triangle Kb}{c^2+b^2K^2}i, \text{ and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}+\frac{2\triangle Kc}{c^2+b^2K^2}i.$$

If P and Q denote the imaginary points of intersection, we see that their coordinates are conjugates. These points are called "conjugués harmoniques" with respect to B and C, by M. Chasles.

It is evident that by giving different values to K an infinite number of such points can be found.

III. **Solution by the PROPOSER.**

The roots of $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$ will be harmonic if $ac'+a'c-2bb'=0$ (see Scott's Geometry, page 45).

Let $x^2=p^2$ give the points A and B. Let $x=OM=K<(OB=p)$ be midway between the other points, P and Q. The equation giving P and Q is

$$a'x^2+2b'x+c'=0, \text{ with the conditions } \frac{b'}{a'}=-K, \text{ and } c'-p^2a'=0,$$

$$\text{or } x^2-2Kx+p^2=0.$$ [redacted]

But since $K<p$, $K^2-p^2<0$, the roots of this equation are imaginary, and since there are an indefinite number of values for $K<p$, there will be an indefinite number of pairs of imaginary points on the line harmonic with the given real pair. (Scott's Geometry, page 45.)

Solved in a similar manner by *G. B. M. ZERR.*

PROBLEMS.

63. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.**

Prove, analytically :—A rectangular hyperbola cannot be cut from a right circular cone unless the angle at its vertex is greater than a right angle.

64. **Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society and Treasurer of Grant County, Marion, Indiana.**

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A particle is placed within a thin cylindrical shell without ends. Find the resultant attraction, the cylinder being composed of matter attracting according to the laws of nature.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $r=2a\sin\theta$ be the equation to the cylinder, so that the origin is in the surface of the cylinder at one end, then $y=r\sin\theta=2a\sin^2\theta$, $z=r\cos\theta=2a\sin\theta\cos\theta$, l=length of cylinder, (x, y, z) coordinates of any point in the shell, ρ=density, k=thickness of shell. It is always possible to take the axes of coordinates so that the particle will lie in the plane of the axis of y; let (m, n, o) be the coordinates of the particle, mass unity.

$$ds=2adxd\theta,\ p=\sqrt{(x-m)^2+(y-n)^2+z^2}=\sqrt{(x-m)^2+n^2-4a(n-a)\sin^2\theta},\ n>a.$$

This will give attraction for all possible positions of the particle. For $n<a$,

$$p=\sqrt{(x-m)^2+(2a-n)^2-4a(a-n)\sin^2(\tfrac{1}{2}\pi-\theta)},$$

and the solution would be the same as for $n>a$.

$$\text{Let } \frac{4a(n-a)}{m^2+n^2}=b^2,\quad \frac{4a(n-a)}{(l-m)^2+n^2}=c^2,\quad \frac{4a(n-a)}{n^2}=d.$$

Resolving the attractions parallel to the axes, we easily get

$$X=2a\rho k\int_0^\pi\int_0^l\frac{(x-m)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}$$

$$=2a\rho k\int_0^\pi\left\{\frac{1}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}-\frac{1}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta$$

$$=\frac{2a\rho k}{\sqrt{m^2+n^2}}E_0^\pi(b,\theta)-\frac{2a\rho k}{\sqrt{(l-m)^2+n^2}}E_0^\pi(c,\theta).$$

$$Y=2a\rho k\int_0^\pi\int_0^l\frac{(2a\sin^2\theta-n)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}},$$

$$=2a\rho k\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{(2a\sin^2\theta-n)d\theta}{n^2-4a(n-a)\sin^2\theta}$$

$$=\frac{a\rho kn(2a-n)}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{d\theta}{n^2-4a(n-a)\sin^2\theta}$$

$$-\frac{a\rho k}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta,$$

$$\therefore\ Y=\frac{a\rho kn(2a-n)}{n^2(n-a)}\left[\frac{l-m}{\sqrt{(l-m)^2+n^2}}\Pi_0^\pi(-d,c,\theta)\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2}}\Pi_0^\pi(-d,b,\theta)\right]$$

$$-\frac{a\rho k(l-m)}{(n-a)\sqrt{(l-m)^2+n^2}}E_0^\pi(c,\theta)-\frac{a\rho km}{(n-a)\sqrt{m^2+n^2}}E_0^\pi(b,\theta).$$

$$Z=2a\rho k\int_0^l\int_0^\pi\frac{2a\sin\theta\cos\theta dxd\theta}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}=0.$$

F=resultant attraction $=\sqrt{X^2+Y^2+Z^2}$.

When $n=a$, the particle is on the axis of the cylinder, then

$$F=X=2\pi a\rho k\left\{\frac{1}{\sqrt{m^2+a^2}}-\frac{1}{\sqrt{(l-m)^2+a^2}}\right\}.$$

When $m=\frac{1}{2}l$, the particle is at the center of the cylinder, and $F=0$.

When $m=l, n=a, F=2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $m=0,\ n=a,\ F=-2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $n=2a$ the particle is on the surface of the cylinder,

$$\text{then } b^2=\frac{4a^2}{m^2+4a^2},\quad c^2=\frac{4a^2}{(l-m)^2+4a^2},\ d=1.$$

$\therefore$ The elliptic function of the third order in Y disappears.

PROBLEMS.

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Find the time of vibration of a particle *slightly* displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

43. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytic Mechanics, page 54.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by W. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

In a parallelogram, sides a and b, diagonals c and d, $2a^2+2b^2=c^2+d^2$. Find all the parallelograms, not rectangles, whose sides and diagonals are rational.

Examples:	a	b	c	d
	4	7	9	7
	16	7	21	13
	8	9	13	11
	8	11	17	9

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

By means of the sides and diagonals we can form, in each parallelogram, two different triangles, the sides of one being a, b, and c, and of the other, a, b, and d.

Take the triangle, sides a, b, and c and put $a=n$, $b=n+p$, and $c=2n\pm q$. From the relations of the sum and the difference of any two sides to the third side, we have the following conditions: $p\mp q>0$ and $p\mp q<2n$. For $p-q$, $c=2n+q$; and for $p+q$, $c=2n-q$.

The *median* upon c is $1/2\sqrt{2(a^2+b^2)-c^2}$. But as the diagonals of a parallelogram bisect each other, this median equals $\frac{1}{2}d$. Whence $d^2=2(a^2+b^2)-c^2=4n(p\mp q)+2p^2-q^2$.

Then $n=\dfrac{d^2-2p^2+q^2}{4(p\mp q)}$. But we have found that $2n>p\mp q$. Therefore

$\dfrac{d^2-2p^2+q^2}{2(p\mp q)}>p\mp q$. Whence $d>2p\mp q$.

Put $d=2p\mp q+t$. Then $a=n=\dfrac{(2p\mp q+t)^2-2p^2+q^2}{4(p\mp q)}$;

$b=n+p=\dfrac{(2p\mp q+t)^2+(2p\mp q)^2-2p^2}{4(p\mp q)}$;

and $c=2n\pm q=\dfrac{(2p\mp q+t)^2-(p\mp q)^2-p^2}{2(p\mp q)}$,

in which p, q, and t are any integers. p and q may also be *zero*, but only one of them in the same operation. When $p=q$ and when $q>p$, we use q only as *positive*, $[+q]$; but when $p>q$, we can use q as both *positive* and *negative*.

When numerical values, assigned to p, q, and t, render a and b or a, b, and c fractional, integral results are obtained by multiplying a, b, c, and d by the least common denominator of the fractions.

Examples:—(1). Put $p=2$, $q=1$, and $t=2$. Then, for $p+q$, $a=7/2$, $b=11/2$, $c=6$, and $d=7$; or in integers, 7, 11, 12, and 14.

(2). Put $p=3$, $q=1$, and $t=2$. Then $a=4$, $b=7$, $c=7$, and $d=9$. Also $a=4$, $b=7$, $c=9$, and $d=7$.

For $p-q$, $a=9/2$, $b=13/2$, $c=10$, $d=5$; or in integers, 9, 13, 20 and 10.

When $q=0$, or when $c=2a$, we have $a=[(2p+t)^2-2p^2]/4p$, $b=[(2p+t)^2+2p^2]/4p$, $c=[(2p+t)^2-2p^2]/2p$, and $d=2p+t$.

Examples:—(1). Put $p=1$, and $t=2$. Then $a=7/2$, $b=9/2$, $c=7$, and $d=4$; or in integers, 7, 9, 14, and 8.

(2). Put $p=t=2$. Then $a=7/2$, $b=11/2$, $c=7$, and $d=6$; or in integers, 7, 11, 14, and 12.

When $p=0$, or when $a=b$, we have $a=[(t\mp q)^2+q^2]/4q=b$,

$c=[(t\mp q)^2-q^2]/2q$, and $d=t\mp q$; or, in integral form, $a=b=(t\mp q)^2+q^2$, $c=2t(t\mp 2q)$, and $d=4q(t\mp q)$.

Examples:—(1). Put $t=q=1$. Then $a=b=5$, $c=6$, and $d=8$.

(2). Put $t=3$ and $q=1$. Then $a=b=17$, $c=30$, and $d=16$. Also $a=b=5$, $c=6$, and $d=8$.

When $q=p$, we have $a=[(3p+t)^2-p^2]/8p$, $b=[(3p+t)^2+7p^2]/8p$, $c=[(3p+t)^2-5p^2]/4p$, and $d=3p+t$.

When $t=q=p$, we have, in integral form, $a=15p$, $b=23p$, $c=22p$, and $d=32p$.

Thus we continue making general values for a, b, c, and d, under a number of other conditions; as, $t=q$; $t=p$; $t=2q=2p$, etc.

43. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the series of integral numbers in which the sum of any two consecutive terms is the square of their difference.

I. Solution by J. H. DRUMMOND, LL. D., Portland, Maine, and the PROPOSER.

Let x and $x+m$ be two consecutive numbers. Then we have $2x+m=m^2$, and $x=m(m-1)/2$, and $x+m=m(m+1)/2$. But $m(m+1)/2$ is the sum of the terms in the series $1+2+3+4$ ....$+m$. Hence the m^{th} term of the series required is the sum of m terms of this series, and we have 1, 3, 6, 10, 15, ...$m(m-1)/2$.

II. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland; and BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

By the conditions we must have $x+y=(x-y)^2$, x and y representing two consecutive terms in the series. Solving as a quadratic in x, we have $x=(2y+1)/2\pm\sqrt{(8y+1)/4}$. Hence $8y+1$ must be a square.

When $y=1$, $8y+1=3^2$, $x=3$;

$y=3$, $8y+1=5^2$, $x=6$;

$y=6$, $8y+1=7^2$, $x=10$;

and the series is, 1, 3, 6, 10, 15, 21, 28, 36, 45, etc., or the system of *triangular* numbers as set forth in Pascal's Triangle.

Also solved by *A. H. HOLMES, E. W. MORRELL, H. C. WILKES*, and *G. B. M. ZERR*.

44. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The hypotenuse of a right-angled triangle ABC, right-angled at A, is extended equally at both extremities so that $BE=CD$. Draw AD and AE. Find integral values for all the lines in the figure thus made.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem. Then draw BF equal and parallel to AC, and draw CF, AF, EF, and DF. Then will $ABFC$ be a rectangle; and the diagonals BC and AF are equal.

It is also evident that $AE=DF$ and $AD=EF$. Whence $AEFD$ is an ob-

lique-angled parallelogram, or rhomboid, of which AE and AD are the sides, and ED and AF the diagonals.

Let $x=BE=CD$, and put $a=AD$, $b=AE$, $c=ED$, and $d=AF=BC$, taking $c>d$. Then $2x+d=c$, and $x=(c-d)/2$. If $d>c$, AE and AD fall inside of AB and AC, and the hypotenuse BC would be *contracted* instead of *extended*.

We now find integral values for a, b, c, and d. This has been done in the solution of No. 42, in this issue, and need not be reproduced here.

By this process we find integral values for all the lines except the two legs, AB and AC, of the right-angled triangle. By means of the median and the perpendicular upon BC, we readily find

$$\overline{AB}^2=d[4b^2-(c-d)^2]/4c \text{ and } \overline{AC}^2=d[4a^2-(c-d)^2]/4c.$$

Now, if these expressions can be rendered squares, without destroying the relations of a, b, c, and d, AB and AC will also be rational and integral. But I have not yet succeeded in accomplishing this. We shall now illustrate by means of a few examples.

From Diophantine problem No. 42, take the set of values, $a=4$, $b=7$, $c=9$, and $d=7$. Then $2x+7=9$; whence $x=1$. $\therefore$ $AD=4$, $AE=7$, $ED=9$, $BC=AF=7$, $BE=DC=1$, $\overline{AB}^2=112/3$, and $\overline{AC}^2=35/3$.

Take the set of values, $a=8$, $b=11$, $c=17$, and $d=9$. Then $2x+9=17$; and $x=4$. Also $\overline{AB}^2=945/17$, and $\overline{AC}^2=432/17$.

Partial solutions also received from *J. H. DRUMMOND*, *A. H. BELL*, and the *PROPOSER*.

PROBLEMS.

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

AVERAGE AND PROBABILITY.

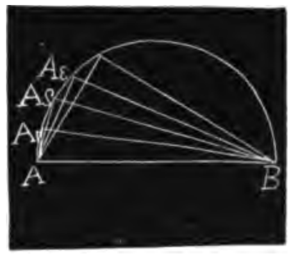
Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

REPLY TO THE REPLIES TO MY "NOTE ON AVERAGE AND PROBABILITY."

BY ARTEMAS MARTIN, LL. D., U. S. COAST AND GEODETIC SURVEY OFFICE, WASHINGTON, D. C.

I WISH to say first that I reaffirm all that I stated on pages 370 and 371, Vol. II., No. 12, and then peoceed to consider the replies of the Repliers.

I. Professor Zerr starts out with the statement that "The problem that gives the result $\frac{1}{8}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$." This is superfluous information; I had clearly set forth that fact in my "Note." But the truth of the next sentence, "In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states," I do not admit, and will proceed to prove its falsity.



Let $AB=a$, the given hypotenuse, which shall remain *fixed*.

Draw A_1B, A_2B, A_3B, A_4B, and so on, the sides AA_1, AA_2, AA_3, AA_4, etc., increasing uniformly from 0 towards a, the consecutive differences, AA_2-AA_1, AA_3-AA_2, AA_4-AA_3, etc., being all equal to each other, and each difference less than any assignable quantity. Thus will be had *all possible* right-angled triangles having the hypotenuse a, and, as I stated on page 371, the right angles *are* all situated on the semi-circumference whose diameter is the given hypotenuse a; but they (the right angles) are not uniformly distributed on this semicircumference because the chords AA_1, AA_2, AA_3, AA_4, etc., increase (or vary) uniformly and therefore their arcs can not increase (or vary) uniformly.

Professor Zerr continues: "The problem [the one that gives the result $\frac{1}{8}a^2$] is as follows: 'Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another'." With all due deference to Professor Zerr, I beg leave to say that I have *not* conceived the triangles to be generated in any such way, as I have clearly shown by the diagram above.

The remainder of Professor Zerr's "Note" does not require considering as it has nothing to do with the matter in hand.

II. I discard the "tail" in italics Professor Matz has appended to the problem; it is not needed to "fly the kite."

I will take up his third and fourth paragraphs. In his third paragraph he says that I, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignore* an infinitude of right-angled tri-

angles. Now if Professor Matz can *prove* that there *are any* right-angled triangles having the hypotenuse a besides those obtained by varying one leg uniformly from 0 to a, I—would like to see the proof. How *can* there *be* any other triangles, if we have a leg for *every possible* value from 0 to a?

III. I will pass over the first and second paragraphs of the Editor's "Reply." In regard to the third paragraph I deny that any triangles *can* be interpolated, and demand proof. If one leg takes *all possible* values from 0 to a, every triangle has been included and there *can not* be any other.

IV. My solution, which I desire to reproduce here, is as follows:

Let x denote one leg of any one of the triangles, then $\sqrt{(a^2-x^2)}$ will denote the other leg. The area of this triangle is $\frac{1}{2}x\sqrt{(a^2-x^2)}$, and the true average of this is

$$\int_0^a \tfrac{1}{2}xdx\sqrt{(a^2-x^2)} \div \int_0^a dx, =\tfrac{1}{6}a^2.$$

V. I think I have considered and fully refuted every objection that has been raised against my solution.

Correction.—Vol. II., page 371, for "p. 82" read p. 282.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude; taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

II. Corrected solution by JOHN M. ARNOLD, Crompton, Rhode Island; and Prof. G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β=altitude, h, h_1 the hour angles.

This event can happen only when Antares is west and Fomalhout east of the meridian.

$$\therefore \left.\begin{aligned}\sin\beta&=\sin\lambda\sin\delta+\cos\lambda\cos\lambda\cos h\\&=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(1).$$

$$\alpha-h=\alpha_1+h_1, \text{ or } h+h_1=\alpha-\alpha_1 \dots\dots\dots\dots\dots(2).$$

But $\lambda=40^\circ$, $\alpha=343^\circ$, $\alpha_1=245^\circ\ 45'$. $\delta=-30^\circ\ 12'$, $\delta_1=-26^\circ\ 12'$.

$\therefore$ $662065\cos h-687337\cos h_1=39538\dots\dots\dots\dots\dots(3).$

$$\cos(h+h_1)=\cos 97^\circ\ 15'=-.12620\dots\dots\dots\dots\dots(4).$$

Let $\cos h=x$, $\cos h_1=y$. From (4) $y=-.12620x\pm.992005\sqrt{1-x^2}$. This in (3) gives, $748806.9294x\mp681841.7407\sqrt{1-x^2}=39538$.

$\therefore$ $x^2-.057736x=.451771$. $\therefore$ $x=.701626$ or $-.643890$.

$\therefore$ $h=45^\circ\ 26'\ 31''$ or $130^\circ\ 4'\ 57''$. The first value of h gives h_1 positive.

$\therefore$ $h=3$ hours, 1 minute, 46 seconds.

$\therefore$ sidereal time$=\alpha-h=19$ hours, 50 minutes, 14 seconds.

37. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A gentleman owned and lived in the center, R, of a rectangular tract of land whose diagonal, D, was 350 rods, dividing the tract into two equal right-angled triangles, in each of which is inscribed the largest square field, F and F_1, possible; the north and south boundary lines of the two square fields being extended and joined formed a little rectangular lot, R, in the center around the residence. The difference in the area of the *entire rectangular* tract and the *sum* of the areas of the two square fields, F, F_1, is $187\frac{1}{2}$ acres. Give the dimensions and area of the entire tract, and one of the square fields, F or F_1.

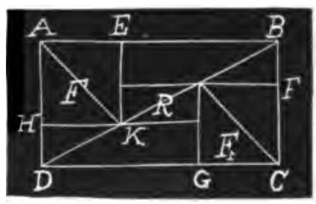
I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $AB=a$, $AD=b$, $AH=x$. $\therefore$ $a^2+b^2=122500\dots\dots\dots(1).$

$ab-2x^2=187\frac{1}{2}$ acres$=30000$ square rods...(2).

$ax+bx=ab$ .... (3),

from triangles BAD and BEK.



From (3) $x^2(a^2+2ab+b^2)=a^2b^2$.........(4).

(1) and (2) in (4) gives $62500x^2=900000000$.

$\therefore$ $x^2=14400$ square rods$=90$ acres.

$\therefore$ $x=120$ rods.

$\therefore$ $ab=58800$ square rods$=367$ acres.

$\therefore$ $a+b=490$ rods. $a-b=70$ rods. $\therefore$ $a=280$, $b=210$.

II. Solution by ISAAC L. BEVERAGE, Monterey, Virginia.

If $a=AB$ and $b=AD$, then ab=area of entire farm. Now $ab/(a+b)=AH$, since it is the side of an inscribed square of a triangle.

$\therefore$ $[ab/(a+b)]^2$=the area of F or F_1. Hence, we readily obtain,

$$ab-2[ab/(a+b)]^2=187\tfrac{1}{2}\times160\dots\dots\dots(1),$$

$$\text{and } \sqrt{a^2+b^2}=350\dots\dots\dots(2).$$

Whence $a=280$ rods, and $b=210$ rods; also $ab=58800$ square rods $=367\frac{1}{2}$ acres. $\therefore ab/(a+b)=120$ rods, and $[ab/(a+b)]^2=14400$ square rods $=90$ acres.

III. **Solution by A. H. BELL, Box 184, Hillsboro, Illinois.**

Let 350 rods $=87\frac{1}{2}$ chains $=2a$, $DK=a-y$ and $BK=a+y$. Also, $187\frac{1}{2}$ acres $=1875$ square chains $=b^2$, and side of square $=x$; then $DG=EB=\sqrt{(a+y)^2-x^2}$, $DH=BF=\sqrt{(a-y)^2-x^2}$,

$$(\sqrt{(a+y)^2-x^2}+x)^2+(\sqrt{(a-y)^2-x^2}+x)^2=4a^2 \quad \ldots\ldots\ldots\ldots(1).$$

Plainly, $$x\sqrt{(a+y)^2-x^2}+x\sqrt{(a-y)^2-x^2}=b^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

(2)×2, and subtracted from (1), when expanded, $y^2=a^2-b^2 \ldots\ldots\ldots(3)$.

$a+y : a-y :: \sqrt{(a+y)^2-x^2} : x. \quad \therefore x^2=(a^2-y^2)^2/2(a^2+y^2) \ldots(4)$.

Substituting values, $y=6.25$ chains, $x^2=90$ acres, $x=30$ chains, $EB=DG=40$ chains, $BF=DH=22.5$ chains, $AB=DC=70$ chains, $AD=BC=52\frac{1}{2}$ chains, $DC\times AD=367\frac{1}{2}$ acres, in the rectangle.

Also solved by *P. S. BERG*, *A. H. HOLMES*, and *B. F. YANNEY*.

PROBLEMS.

45. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School Lawrenceville, New Jersey.

Required several numbers each of which, divided by 10 leaves a remainder 9; by 9 leaves 8; by 8 leaves 7; by 7 leaves 6, and so on. Also find the least such number which, when divided by 28 leaves 27; by 27 leaves 26; by 26 leaves 25; by 25 leaves 24, *et cetera ad unum*.

46. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The base BC of the triangle ABC is $2c$, the sum of the two sides, AB and BC, is $2a$. BP is always perpendicular to AB and cuts AC in P. What is the locus of the point P?

47. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, New York.

In longitude 75 degrees west of Greenwich, latitude 43 degrees, 30 minutes north on January 1, 1895, at 3 o'clock A. M., local time. What points of the ecliptic were then rising, setting and on the meridian? Any other necessary data may be taken from an ephemeris.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In case of *mischance*, with what force would the cow, weighing $w=700$ pounds, jumping over the moon, have struck Her Lunar Majesty in the face?

EDITORIALS.

Our valued contributor, Sylvester Robbins, who is visiting in Southern Ohio, made Prof. William Hoover a pleasant call a few days ago.

Professors W. W. Beman and D. E. Smith are preparing a translation of Klein's Vorträge über ausgewählte Fragen der Elementargeometrie. It will be issued during the winter by Ginn & Co.

We are grieved to record the death of our valued contributor and subscriber, Prof. H. A. Newton, of Yale University, whose death occured August 12th. In a future number of the MONTHLY will appear a biographical sketch of his life, by his colleague, Prof. A. W. Phillips.

The friends of Drury College will be pained to learn of the death of a member of its Faculty, Prof. William J. Whitney, of the Department of History, whose death, caused by typhoid fever, occurred on September 26th, at the home of his father, near Findley's Lake, New York. His broad scholarship, his accurate judgment, and the fine qualities of his character made him a great favorite among the Faculty and students of the College. Professor Whitney was a most intimate and helpful friend of Editor Finkel, and in his death we sustain a great loss.

BOOKS.

Elements of Plane and Spherical Trigonometry, A Text-book for Colleges and Schools. By Edwin S. Crawley, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania. Second edition, revised and enlarged. 8vo. Cloth, 178 pages. Price, $1.00. Published by the Author, Philadelphia, Penn.

This book contains all that is needed on the subject of Trigonometry in our best colleges. The author has omitted nothing that is necessary in studying the branches of Mathematics following Trigonometry. Such important subjects as De Moivre's Theorem, Hyperbolic Functions, Theorems relating to the escribed circles and Brocard's points are concisely treated. The book is very beautifully printed, and substantially bound in cloth. We do not hesitate to recommend this book to teachers and students desiring a good text on the subject treated. B. F. F.

Higher Mathematics. A Text-book for Classical and Engineering Colleges. Edited by Mansfield Merriman, Professor of Civil Engineering in Lehigh University, and Robert S. Woodward, Professor of Mechanics in Columbia University. Large 8vo., 576 pages. Price, $5.00. New York: John Wiley & Sons.

This volume is designed especially for the use of Junior and Senior Classes in Colleges and Technical Schools, but it is equally well adapted to the use of advanced students

and readers of Mathematics generally. The editors have called to their assistance the best mathematicians in the country, and thus given the book weight of authority never before given an American Mathematical Text-book. The book contains a concise treatment of the following subjects, not commonly found in text-books but upon which lectures are now given in our best classical and technical institutions:

Chapter I. The Solution of Equations, by Mansfield Merriman, Professor of Civil Engineering in Lehigh University; Chapter II. Determinants, by Laenas Gifford Weld, Professor of Mathematics in State University of Iowa; Chapter III. Projective Geometry, by George Bruce Halsted, Professor of Mathematics in the University of Texas; Chapter IV. Hyperbolic Functions, by James McMahon, Associate Professor of Mathematics in Cornell University; Chapter V. Harmonic Functions, by Professor William E. Byerly, Professor of Mathematics in Harvard University; Chapter VI. Functions of a Complex Variable, by Thomas S. Fiske, Adjunct Professor of Mathematics in Columbia University; Chapter VII. Differential Equations, by W. Woolsey Johnson, Professor of Mathematics in the U. S. Naval Academy; Chapter VIII. Grassmann's Space Analysis, by Edward W. Hyde, Professor of Mathematics in the University of Cincinnati; Chapter IX. Vector Analysis and Quaternions, by Alexander Macfarlane, Lecturer in Civil Engineering in Lehigh University; Chapter X. Probabilities and Theory of Errors, by Robert S. Woodward, Professor of Mechanics in Columbia University; Chapter XI. History of Modern Mathematics, by David Eugene Smith, Professor of Mathematics in Michigan State Normal School.

It is to be hoped that all classical colleges and other institutions of learning that have no provision for mathematical study in the Junior and Senior years will so arrange the course of study that the Higher Mathematics as here presented may be pursued during the last two years of college work, so that the student, during these years, may not be deprived of the rigid discipline of mind and the culture derived from its study. B. F. F.

Elementary Algebra. By H. S. Hall, M. A., and S. R. Knight, B. A. Revised and Enlarged for the use of American Schools by F. L. Sevenoak, A. M., Assistant Principal of the Academic Department, Stevens Institute of Technology. 8vo. Cloth and Leather Back. 416 pages. Price, $1.10. New York: Macmillan & Co.

Only words of commendation can be said of this book. The complete and accurate treatment of each subject, the abundant illustrations, the scientific arrangement of the subjects, go to make up all that could be desired in a good text-book. This book together with the author's Higher Algebra, makes a very exhaustive course in Algebra. B. F. F.

Euclidian Geometry. By J. A. Gillet, Professor in New York Normal College. 8vo. Cloth and Leather Back. 436 pages. New York: Henry Holt & Co.

This book, as its name implies, reverts to the purely geometric methods of Euclid. The author maintains sharply throughout the work, the distinction between the processes of pure geometry on the one hand and those of arithmetic and algebra on the other. The author says, "Euclidian Geometry bears to modern geometry very much the same relation that arithmetic bears to algebra. Its theorems are less general and it admits of positive magnitude only. For this reason its simple and rigorously logical methods can never be replaced by those of synthetic geometry, either as a factor in general education or as a foundation for advanced study." We can not agree with the Author in his last statement. It has been our experience in teaching geometry that the boy or girl, who studies geometry for the mental discipline it gives him and not merely for grades, feels better satisfied when he has demonstrated a proposition in its entirety, than he does when he has demonstrated one which he feels must be enlarged, as he advances in the study of Mathematics, to satisfy all cases. However, there is much in the book to commend it favorably to teachers. B. F. F.

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MATHEMATICAL INFINITY AND THE DIFFERENTIAL.

By FRANKLIN A. BECHER, Milwaukee, Wisconsin.

Mathematics, as defined by the great mathematician, Benjamin Pierce, is the science which draws necessary conclusions. In its broadest sense, it deals with conceptions from which necessary conclusions are drawn. A mathematical conception is any conception which, by means of a finite number of specified elements, is precisely and completely defined and determined. To denote the dependence of a mathematical conception on its elements, the word "manifoldness," introduced by Riemann, has been recently adopted. Manifoldness may be looked upon as the genus, and function, as the species. This conception reaches down to the very foundation of mathematical concepts and principles. It is the central idea from which the whole field and range of the mathematical sciences may be surveyed. Time, space, and numbers are included in the notion, manifoldness.

Manifoldness may be defined according to Dr. Cantor as being in general every *muchness* or complexity which may be conceived as a unit, or a number of objects, conceptions, or elements which are united in one law or system.

Manifoldness may be devided into discrete and continuous. Proceeding with the conception of whole numbers as it is obtained by counting and extending the same by means of the divisibility of numbers so as to include the conception of the rational system of numbers, we have one of the elements which enter into the conception of a discrete manifoldness. The irrational system of numbers is included in the conception of continuous manifoldness. This must not be considered as an inherent division, for it is well to note here that in the higher analysis, in one instance and for one purpose, a conception may be considered as a discrete manifoldness and for another purpose as a continuous manifoldness.

The three laws of operation, i. e., the law of commutation, of association, and of distribution, hold good in all forms of calculation, whether discrets or continuous manifoldness. From these laws, the four processes, addition, subtraction, multiplication, and division are derived.

Number, in all its forms, whether finite or infinite, rational or irrational, constant or variable, continuous or discontinuous, is included as one of the elements of manifoldness.

We will now consider number with special reference to its limits, infinity and zero, by the introduction of the conception of variability, of continuity, and of the differential.

By means of an unlimited continuous series of rational numbers, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\ldots \alpha_n$, whose terms have the property that there be given to every number δ, however small, a place n, from which the difference of all succeeding numbers remains smaller than δ, we define a definite number which is called a limit of this series. The creation of this conception admits of a comparison of rational numbers with respect to their magnitude. If all the numbers of the series differ after the place α_n by less than the number δ, then the limit is a number which lies between $\alpha_n - \delta$ and $\alpha_n + \delta$, which, because δ may be chosen as small as we please, can be expressed by a rational number as near as we please.

The totality of all numbers of an interval, for example, 0 to 1, consists not only of all numbers between 0 and 1, but of the totality of all numbers which may be interpolated between the limiting values of the defined series of numbers. This totality we dominate the aggregate or inclusive of the continuous series of numbers.

It is apparent that the conception of a limit of variability and of continuity have their root in irrationality. The two conceptions attached to a limit are in their nature entirely different. In the first instance, a limit may be defined as a limit of a variable, a limitless increasing or decreasing; in the second instance a limit means that which exceeds all limits of measurable number, either because it possesses no magnitude or because the amount or extent would not be exhausted by means of all the series of all numbers though they were being perfected. In the first case, we deal with variable numbers ; in the second case with the conception of the absolute value of the numbers derived from the formation of zero and the conception of infinity.

Zero and infinity are the limits of the natural series of numbers. They are derived in the same manner as the rational series of numbers. Infinity is the result of unlimited addition of unity or other positive numbers, the unlimited multiplication of whole numbers except unity. Zero is derived from the subtraction of two equal numbers. These are the fundamental conceptions of zero and infinity as derived in the lower analysis. It is evident from the different ways in which each of these symbols are derived that they have different meanings attached to them. We may note here that every problem carries inherently with it its solution. The meaning of every symbol depends upon its origin, deriv-

ation and relation. In different problems they may have different meanings. Symbols of quantity, like words, have different definitions, and these are to be determined according to the nature of the problem and their relation to other symbols.

In the higher analysis, the conceptions of infinity and zero present themselves more systematically in the developement of infinite series, infinite products, infinite continued fractions, etc. An infinite number is defined as a variable number, whose absolute value is conceived as being in an unlimited state of increasing or decreasing. In the first instance it is called infinitely large, in the second, infinitely small. The addition of a number of infinitely large or infinitely small numbers will produce an infinitely large or small number. The difference between two infinitely large or infinitely small numbers, where either or both are equal, is zero. However, if they are not equal, the difference can never be a finite number, but must always be an infinite number; otherwise an infinite number would be increased or decreased by a finite number, which is without meaning.

The addition and subtraction of infinite numbers can never produce anything else than infinite numbers or, in a particular case, zero. Again the multiplication or division of an infinite number by a finite number or by infinity will produce like results, i. e., it may be merely an indicated operation, not a completed operation. For instance;

$$2\times\infty=2\,\infty\,;\; n\times\infty=n\,\infty\,;\; \infty\times\infty=\infty^{2},\ \&c.$$

It is apparent that the unlimited number of changes which may be thought of under the conception of infinity as defined here are extraordinarily manifold.

If we conceive an infinite number to grow so that it is continuously twice as large as any other infinite number, then the first is derived from the second by multiplying by two or the second by dividing by two. Multiplying an infinite number by another gives us infinity of a higher power or dividing gives us infinity of a lower power. Every change in value of a variable suggests an increment.

There are two kinds of conceptions associated with increments: the one is that the absolute value of the increment is capable of divisibility. The conditions, however, of which are such that it cannot be conceived smaller. The other is that the absolute value is incapable of divisibility. In the first instance the increments are of such a nature that the variables must stand in a certain relation to one another and if this takes place they are known in higher mathematics as differentials; those of the second kind are of that nature that they do not stand in any relation to one another; these may be called absolute elements of quantity.

Thus, if we pass from one interval of value of a variable to another, there lies between the two a difference which must be considered as possessing quantity, but does not possess the capability of divisibility and this difference in in-

crements is an element of quantity. Increments and differentials are not identical. The former are vested with quantity, while the latter are vested with quality, i. e., they are formal in their nature.

Milwaukee, Wisconsin, September, 1896.

A PROBLEM IN ASTRONOMY.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

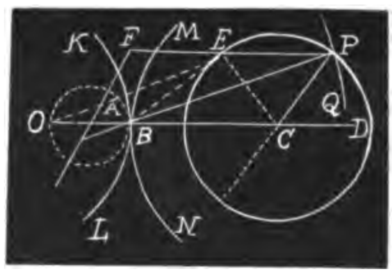
To find the Distance from the Earth to the Sun knowing the distance from an Inferior planet to the Sun supposing the planets to describe circles around the Sun.

Let P be a point on the epicyclic curve PQ, OC the radius of the deferent, CP the radius of the epicycle. Let $CO : CB=n : 1$. $\therefore CB=\frac{CO}{n}$.

Then $BO=CO-CB=CO-\frac{CO}{n}=CO(1-\frac{1}{n})$.

Now the angular velocity $=\frac{\text{transverse velocity}}{\text{radius vector}}$.

The transverse velocity of E is represented by EA in magnitude, and in direction by BA. Let the linear velocity of the mean point (C) be V, the linear velocity of the moving point in the epicycle is



$$nV.\frac{PC}{CO}.$$

$\therefore$ The transverse velocity=

$$\frac{nV}{CO}.EB\cos BEO;$$

$$\text{but } \cos BEO=\frac{(BE)^2+(EO)^2-(BO)^2}{2BE.EO}.$$

$\therefore$ Transverse velocity $=\frac{nV}{CO}.\frac{BE^2+EO^2-BO^2}{2EO}$. Also radius vector $=EO$.

$$\therefore \text{Angular velocity}=\frac{nV}{CO}.\frac{BE^2+EO^2-BO^2}{2EO^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Let the deferential angle$=\theta$, then angle $ECD=(n-1)\theta$.

$\therefore\ BE^2=CE^2+CB^2+2CE.CB\cos(n-1)\theta$

$$=CE^2+\frac{CO^2}{n^2}+2CE.\frac{CO}{n}\cos(n-1)\theta$$

$$EO^2=CO^2+CE^2+2CO.CE\cos(n-1)\theta$$

$$BO^2=CO^2\left(1-\frac{1}{n}\right)^2.$$

Substituting these values in (A) we get

$$\text{Angular velocity}=\frac{V}{CO}\cdot\frac{CO^2+nCE^2+(n+1)CO.CE\cos(n-1)\theta}{CO^2+CE^2+2CO.CE\cos(n-1)\theta}.$$

Now in inferior conjunction, if the moving planet is inferior, $(n-1)\theta=180^\circ$.

$$\therefore\ \text{Angular velocity}=\frac{V}{CO}\cdot\frac{CO^2+nCE^2-(n+1)CO.CE}{CO^2+CE^2-2CO.CE}.$$

Let $CO=R$, $CE=r$.

$$\text{Then Angular velocity}=\frac{V}{R}\cdot\frac{R^2+nr^2-(n+1)R.r}{(R-r)^2}.$$

Now $n=\left(\frac{R}{r}\right)^{\frac{3}{2}}$, also putting $\frac{V}{R}=\omega$.

$$\therefore\ \text{Angular velocity}=\omega.\frac{R^2+\left(\frac{R}{r}\right)^{\frac{3}{2}}r^2-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}R.r}{(R-r)^2}$$

$$=\omega\frac{1+\left(\frac{R}{r}\right)^{\frac{3}{2}}\frac{r^2}{R^2}-\frac{R^{\frac{3}{2}}+r^{\frac{3}{2}}}{r^{\frac{3}{2}}}\cdot\frac{r}{R}}{\left(1-\frac{r}{R}\right)^2}\ \ldots\ldots\ldots\ldots\ldots(B)$$

$$=\omega\frac{\frac{R^2}{r^2}+\left(\frac{R}{r}\right)^{\frac{3}{2}}-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}\frac{R}{r}}{\left(\frac{R}{r}-1\right)^2}\ \ldots\ldots\ldots\ldots(C).$$

Let the distance from the earth to the sun be known to find the distance from the planet to the sun.

Let $\frac{r}{R}=\rho$, then (B) becomes

$$\text{Angular velocity}=\omega.\frac{1+\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}}-\rho}{(1-\rho)^2}=\omega.\frac{(1-\rho)-\rho^{-\frac{1}{2}}(1-\rho)}{(1-\rho)^2}$$

$$=\omega.\frac{1-\rho^{-\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}}.\frac{1-\rho^{\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}+\rho}\ \dots\dots\dots(D).$$

Let the distance from the planet to the sun be known to find the distance from the earth to the sun.

Let $\frac{R}{r}=\rho'$, then (C) becomes

$$\text{Angular velocity}=\omega.\frac{\rho'^{2}+\rho'^{\frac{1}{2}}-\rho'^{\frac{3}{2}}-\rho'}{(\rho'-1)^2}=\omega\frac{\rho'(\rho'-1)-\rho'^{\frac{1}{2}}(\rho'-1)}{(\rho'-1)^2}$$

$$=\omega.\frac{\rho'-\rho'^{\frac{1}{2}}}{\rho'-1}=-\frac{\omega\rho'}{1+\sqrt{\rho'}}\ \dots\dots\dots\dots\dots\dots\dots\dots(E).$$

Case I. A planet transits the sun's disc at such a rate that the sun's diameter S would be traversed in time t. Find the planet's distance from the sun.

Let ρ=planet's distance, unity being the earth's distance, and let ω be the earth's angular velocity around the sun=sun's angular velocity around the earth, and let t' be the time in which the sun in his annual course moves through a distance equal to his own apparent diameter; then $\omega t'=S$. From (D) the planet's angular velocity about the earth$=-\frac{\omega}{\sqrt{\rho}+\rho}$.

∴ That is the planet's retrograde gain on the sun is

$$\frac{\omega}{\sqrt{\rho}+\rho}+\omega=\frac{S}{t}=\frac{\omega t'}{t}.$$

$$\therefore\ \rho+\rho^{\frac{1}{2}}=\frac{t}{t'-t},\quad \therefore\ \rho^{\frac{1}{2}}=\tfrac{1}{2}\Big(\pm\frac{\sqrt{3t+t'}}{\sqrt{t'-t}}-1\Big).$$

$$\therefore\ \rho=\tfrac{1}{2}\left(\frac{t'+t}{t'-t}-\sqrt{\frac{3t+t'}{t'-t}}\right)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1).$$

Case II. If we wish to find the earth's distance knowing the planet's distance, then let the planet's distance be unity and the earth's distance$=\rho'$.

Proceeding the same as before using (E) we get

$$\frac{\omega\rho'}{1+\sqrt{\rho'}}+\omega=\frac{S}{t}=\frac{\omega t'}{t}. \quad \therefore \rho'-\frac{t'-t}{t}\sqrt{\rho'}=\frac{t'-t}{t};$$

$$\therefore \sqrt{\rho'}=\frac{1}{2t}\left\{(t'-t)+\sqrt{t'^2-3t^2+2tt'}\right\}.$$

$$\therefore \rho'=\frac{t'-t}{2t^2}\left\{(t'+t)+\sqrt{t'^2-3t^2+2tt'}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Suppose Venus transits the sun's disc at such a rate that the sun's apparent diameter would be traversed in 7⅓ hours, and at the same time the sun in his annual course moves through a distance equal to his own apparent diameter in 12 hours. Required (1) the distance from Venus to the sun, the earth's distance being unity, and (2) the distance from the earth to the sun, Venus's distance being unity.

Now $t=7\frac{1}{3}$, $t'=12$; hence for first case substituting in (1)

$$\rho=\tfrac{1}{2}\left(\tfrac{29}{7}-\sqrt{\tfrac{51}{7}}\right)=.721824.$$

For the second case substitute in (2)

$$\rho'=\tfrac{7}{484}\{58+\sqrt{1428}\}=1.38538.$$

(The above is suggested in Proctor's Geometry of the Cycloid.)

A PROPOSITION IN DETERMINANTS.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

Theorem.—The product of two numbers, each the sum of four squares, is the sum of eight squares.

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix} \times \begin{vmatrix} \alpha+\beta\sqrt{-1} & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} \end{vmatrix}$$

$$=\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} & 0 \\ c+d\sqrt{-1} & a-b\sqrt{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix} \times(-1) \begin{vmatrix} \alpha+\beta\sqrt{-1} & 0 & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & 0 & \alpha-\beta\sqrt{-1} \\ 0 & 1 & 0 \end{vmatrix}$$

$$=(-1)\begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} & -c+d\sqrt{-1} \\ c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} & a-b\sqrt{-1} \\ -\gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} \\ -c\gamma+d\delta+(c\delta+d\gamma)\sqrt{-1} & c\alpha-d\beta-(c\beta+d\alpha)\sqrt{-1} \end{vmatrix}$$

$$+ \begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} \\ -a\gamma+b\delta+(a\delta+b\gamma)\sqrt{-1} & a\alpha-b\beta-(a\beta+b\alpha)\sqrt{-1} \end{vmatrix}$$

or $(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=(c\alpha-d\beta)^2+(c\beta+d\alpha)^2$
$+(c\gamma-d\delta)^2+(c\delta+d\gamma)^2+(a\alpha-b\beta)^2+(a\beta+b\alpha)^2+(a\gamma-b\delta)^2+(a\delta+b\gamma)^2$.

Euler's Theorem is an easy corollary of this, and *vice-versa*.

University of Mississippi, March, 1896.

A METHOD OF SOLVING QUADRATIC EQUATIONS.

By Prof. HENRY HEATON, M. Sc., Atlantic, Iowa.

Let it be required to solve the equation

$$ax^2+bx+c=0 \qquad\qquad (1).$$

Transposing the middle term we have

$$ax^2+c=-bx \qquad\qquad (2).$$

Squaring, $a^2x^4+2acx^2+c^2=b^2x^2$ .... (3).

Subtracting $4acx^2$, $a^2x^4-2acx^2=(b^2-4ac)x^2$ .... (4).

Extracting the square root, $ax^2-c=\pm(\sqrt{b^2-4ac})x$ .... (5).

Adding equation (2), $2a^2x^2=(-b\pm\sqrt{b^2-4ac})x$ .... (6).

Whence $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let it be required to solve the equation $3x^2-2x=21$.

Transposing $2x$ to the second member and 21 to the first, the equation becomes

$$3x^2-21=2x \qquad\qquad (7).$$

Squaring, $9x^4-126x^2+441=4x^2$ ....(8).

Adding twice $126x^2$, $9x^4+126x^2+441=256x^2$ ....(9).

Extracting the square root, $3x^2+21=\pm16x$ ....(10).

Adding equation (7), $6x^2=18x$ or $-14x$.

$\therefore\ x=3$ or $-2\frac{1}{3}$.
Is this new?

[NOTE.—We do not remember of ever having seen this method. If any of our readers have seen it elsewhere, please let us know. Editor.]

ON THE DOCTRINE OF PARALLELS.

By Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

I desire to enter my protest against any assumption that parallel lines, extended to an infinite distance, do, or do not, intersect. The human mind cannot comprehend the infinite and, therefore, we cannot determine the question. We may use modes of reasoning involving infinite quantities, but we can rely upon the results *only so far as human experience shows that they are correct.* It is true, that a mode of reasoning in such cases. which leads to a result found by human experience to be correct in a particular case, may generally be assumed to be correct in all cases. Without human experience, the proposition that if two objects are moving in the same line in the same direction at different velocities, the one in advance will move over an appreciable space while the other is moving over the space between them and, therefore, that the one can never overtake the other, could never have been successfully denied. I hold that this doctrine applies to much of the discussion of the present day, and some of the propositions I have been able to deny, and old propositions denied I have been able to affirm, because I knew that *human experience had settled the matter.*

Whether Euclid's reasoning was, or was not correct, I have never seen a case in which the result which he reached has not been found to be absolutely correct by human experience.

The quotation which Professor Lyle makes from Lotze (Vol. II. page 375) involves the arrogant assumption that the human mind is infinite in the scope of its reasoning power. Mathematicians, of all men, should not claim that a proposition involving the infinite, cannot be true, because we cannot *comprehend the possibility* of its being true.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60\%$?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and J. F. YOTHERS, Westerville, Ohio.

$m(1+p)=$price at which a quart of pure milk would sell at a profit of $p\%$.

$\frac{m(1+p)}{n}=$number of quarts at n cents sold for $m(1+p)$ cents.

$\therefore \frac{m(1+p)}{n}-1=\frac{m(1+p)-n}{n}=$amount of water added to each quart of milk. Let $m=5$, $n=6$, $p=.60$.

$\therefore \frac{m(1+p)-n}{n}=\frac{1}{3}$. $\therefore$ He adds one quart of water to 3 quarts of milk.

Also solved by *E. R. ROBBINS*, *P. S. BERG*, *F. R. HONEY*.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6%?

I. Solution by P. S. BERG, Larimore, North Dakota; EDWARD R. ROBBINS, Lawrenceville, New Jersey; FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Present worth of \$100 for 2 years at 6%=\$89.28.
Present worth of \$200 for 4 years at 6%=\$161.29.
\$250.57=\$89.28+\$161.29=sum of present worths,
The time required for \$250.57 at 6% to amount to \$300 is the time sought.
Interest of \$250.57 for one year at 6%=\$15.0342.
\$300—\$250.57=\$49.43, interest for the whole term.
Hence time equals 49.43÷15.0342=3.2878 years.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; J. F. YOTHERS, Westerville, Ohio.

The interest on \$100 for 2 years at 6%=\$12.
The interest on \$200 for 4 years at 6%=\$48.
The interest on \$300 for 1 year at 6%=\$18.
(\$12+\$48)÷\$18=\$60÷\$18=3 years, 4 months.

Or, \$100 for 2 years = \$ 200 for 1 year.
\$200 for 4 years = \$ 800 for 1 year.

\$1000 for 1 years.

\$1000 ÷ \$300 = 3 years, 4 months.

PROBLEMS.

67. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth?

68. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $P=68921$, what was it $n=3$ years ago? At this rate of increase, what will the population be $n=3$ years hence?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apollonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

1. In trilinears, the equation to the circumcircle of the triangle of reference is

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta=0\text{.......................}(1).$$

The tangent to this circle at A is

$$b\gamma+c\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation to the symmedian through A is

$$b\gamma-c\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

(2) and (3) are conjugates to $\beta=0$, $\gamma=0$.

2. The circle of Apollonius passes through A and the points of intersection of the internal and external bisectors of the angle A of the triangle of reference with the side BC. The coordinates of the center of this circle are plainly the half sum of those of the intersections of the bisectors with BC, or $\left(0, \frac{ab^2\sin C}{b^2-c^2}, \frac{ac^2\sin B}{b^2-c^2}\right)$. This is the point of intersection of (2) and $\alpha=0$.

3. The coordinates of Grebe's point are proportional to a, b, c.

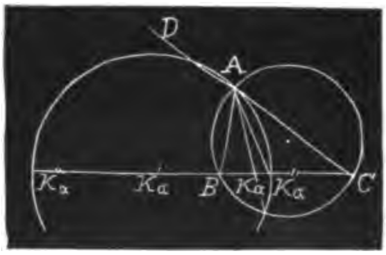
The mid-point of a is $(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B)$, and of the altitude on a, $\left(\frac{\Delta}{a}, \frac{b}{4}\sin 2C, \frac{c}{4}\sin 2B\right)$.

The line through these points is $\alpha\sin(B-C)+\gamma\sin C-\beta\sin B=0$, which is satisfied by $\alpha=a$, $\beta=b$, $\gamma=c$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we get,

(1), equation to AK_a' is, $\beta\sin C+\gamma\sin B=0$, or $\beta c+\gamma b=0$; to AB, $\gamma=0$; to AK_a, $\beta c-\gamma b=0$; to AC, $\beta=0$.



$\therefore$ AK_a', AB, AK_a, AC form a harmonic pencil.

$\therefore$ K_a' and K_a are harmonic conjugate.

(2). Draw AK_a''' bisecting $\angle DAB$.

Then $\angle AK_a'B=\angle K_a'K_a'''A+\angle K_a'AK_a'''=B-C$, and $\angle K_a'AK_a'''=\angle K_a'''AB-\angle K_a'AB=90^\circ-\frac{1}{2}A-C=\frac{1}{2}(B-C)$.

$\therefore$ $\angle K_a'K_a'''A=\angle K_a'AK_a'''$; $\therefore$ $K_a'K_a'''=AK_a'=AK_a''$.

$\therefore$ K_a' is the center of the required circle.

(3). The equation to the straight line through Grebe's point and the mid-point of BC is the same as the equation of the straight line through mid-point of BC and the midpoint of the perpendicular from A on BC, both being $\sin(B-C)\alpha-\sin B\beta+\sin C\gamma=0$.

III. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey, and J. C. CORBIN, Pine Bluff, Arkansas.

(1). Let ABC be $\triangle$; AK_a', the tangent at A; and K_a point where sym-

median meets the side BC. Take $AN=AC$, and $AM=AB$. Then in $\triangle ANM$, AH, the median, is the symmedian of $\triangle ABC$. Lines BC, and MN are antiparallel. Also, since $\measuredangle ABC=\measuredangle BAD$, each being measured by ½arc AC, the lines BC and AK_a' are antiparallel. Wherefore MN is parallel to the tangent line AK_a'.

Now we have a pencil of four rays AB, AH, AC, AK_a' in which one ray AH bisects a line parallel to its conjugate, and included between the other pair of conjugate rays; hence the pencil is harmonic, and any line, as BK_a', drawn across the pencil will cut out an harmonic range $\{BC,\ K_aK_a'\}$. Q. E. D.

(3). Draw ST perpendicular to AC at its middle point S; draw BT, and it is a symmedian line (Halsted: Syn. Geom. §648.), hence it passes through Grebe's point (or Lemoine's Point) K. Now as $A\{BC,\ K_aK_a'\}$ is an harmonic pencil, $\{BR,\ KT\}$ is an harmonic range; whence $S\{BR,\ KT\}$ is an harmonic pencil. Draw altitude BP, and it is $\parallel$ to ray ST, and is therefore bisected by the ray SK, the conjugate of ST. Therefore the line joining the middle point, S, of a side, and the middle point of the altitude to that side passes through Grebe's point.

Q. E. D.

59. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2acxz+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

Solution by the PROPOSER.

The tangent plane at $(x',\ y',\ z')$ to $F=0$ . . . . (1) is

$$(x-x')\frac{dF}{dx'}+(y-y')\frac{dF}{dy'}+(z-z')\frac{dF}{dz'}=0 \quad\ldots\ldots(2).$$

Here $F=a^2x^2+b^2y^2+c^2z^2-2bcyz-2acxz-2abxy$ . . . . (3).

$$\frac{dF}{dx'}=2a(ax'-by'-cz'),\quad \frac{dF}{dy'}=2b(-ax'+by'-cz'),$$

$$\frac{dF}{dz'}=2c(-ax'-by'+cz') \quad\ldots\ldots(4).$$

Then (2) becomes by aid of (3),

$$a(ax'-by'-cz')x+b(-ax'+by'-cz')y+c(-ax'-by'+cz')z=0 \quad\ldots\ldots(5).$$

It may be shown that the condition that

$lx+my+nz=0$ ....(6) cuts $ayz+bxz+cxy=0$....... ... .(7)

in two straight lines including a right angle is $amn+bnl+clm=0$...........(8).

Comparing (5) and (6), $l=a(ax'-by'-cz')$, $m=b(-ax'+by'-cz')$, $n=c(-ax'-by'+cz')$, and (8) becomes

$abc\{a^2x'^2-(by'-cz')^2+b^2y'^2-(cz'-ax')^2+c^2z'^2-(ax'-by')^2\}=0$ ....(9),

an identity by aid of (3).

Also solved by *HENRY HEATON* and *J. SCHEFFER*.

PROBLEMS.

65. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

Prove in a pure geometrical way the following:

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's ellipse) are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with a velocity of V and the wheel rotates with velocity ω, what is the component of force or pressure along the axis if it is turned at an angte α to the direction of the wind assuming the resistance of the wheel in turning to be R?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=projecting area of fans exposed to the wind, in square feet,
V=velocity of wind in feet per second,
H=horse hower of pump,
R=extreme radius of fans in feet,

r=inner radius of fans in feet,

$l=\sqrt{\dfrac{R^2+r^2}{2}}$, =radius of center of percussion, in feet,

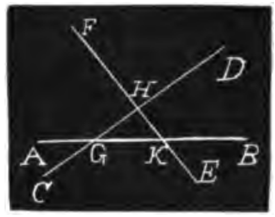
n=number of revolutions of fans per minute,

β=mean angle of fans to the plane of motion.

By Nystrom's Mechanics we get $H=\dfrac{Aln\sin\beta\cos\beta}{1.540000}\left(V-\dfrac{2ln\sin\beta}{19}\right)^2$.

Let AB, CD, FE be the direction of the axis, fans, and wind, respectively.

$\angle HKG=\alpha$, $\angle KGH=\dfrac{\pi}{4}$. $\therefore \angle GKH=\left(\dfrac{3\pi}{4}-\alpha\right)$.



Then $A=60\times150\times\sin\left(\dfrac{3\pi}{4}-\alpha\right)$

$\div144=1\frac{2}{8}{}^5\sin\left(\dfrac{3\pi}{4}-\alpha\right)$.

$n=\omega$, $\beta=\dfrac{\pi}{4}$. $\therefore H=\dfrac{125l\omega\sin\left(\dfrac{3\pi}{4}-\alpha\right)\left(19V-l\omega\sqrt{2}\right)^2}{117040000}$.

$H'=H-R/33000$=effective horse power.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A man weighs 150 pounds; his balloon with all its attachments weighs 500 pounds. What volume of pure hydrogen must be made and put into the balloon so that it will be on the point of ascending with the man? How many kilograms of zinc and of hydrogen sulphate will be used generating the hydrogen? Give volume of hydrogen in cubic feet, given that one litre of hydrogen weighs .0896 grams.

Solution by P. S. BERG, Larimore, North Dakota, and the PROPOSER.

1 grain$=.0022046$ pounds. 1 cubic foot$=28.315$ litres.

Let temperature and pressure be normal.

$\therefore$ 1 cubic foot of hydrogen weighs $28.315\times.0896\times.0022046=.005593123$ pounds.

1 cubic foot of air weighs $28.315\times1.293\times.0022046=.080713261$ pounds.

$\therefore$ The lifting power of 1 cubic foot of hydrogen is .080713261 pounds $-.005593123$ pounds$=.075120138$ pounds.

500 pounds + 150 pounds = 650 pounds.

$650\div.075120138=8652.806$ cubic feet of hydrogen.

$8652.806\times.005593123\div2.2046=21.9524$ kilograms of hydrogen used.

$Zn+H_2SO_4=ZnSO_4+H_2$. $\therefore H_2SO_4 : H_2=x : 21.9524$.

$98 : 2=x : 21.9524$. $\therefore x=1075.6676$ kilograms of hydrogen sulphate.

$Zn : H_2=x : 21.9524$. $65 : 2=x : 21.9524$. $\therefore x=713.453$ kilograms of zinc.

Also solved by *A. P. REED*.

PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We easily find by inspection $U_{x+1}-5U_x=\dfrac{4^{\frac{x+1}{2}}-1}{3}$, or $\dfrac{4^{\frac{x+2}{2}}-1}{3}$, according as x is odd or even. Integrating and reducing, we have

$$U_x=\tfrac{1}{4}[5^x+4\times5^{x-2}+4^2\times5^{x-4}+\text{etc.}-\frac{4^{\frac{x+1}{2}}-1}{3}\text{ or }\frac{4^{\frac{x+2}{2}}-1}{3}].$$

$$\text{Summing, } S_x=\tfrac{1}{16}[5^{x+1}+4\times5^{x-1}+4^2\times5^{x-3}+\text{etc.}-\frac{23\times4^{\frac{x+2}{2}}-12x-47}{9},$$

$$\text{or } \frac{11\times4^{\frac{x+3}{2}}-12x-47}{9}].$$

Putting $x=20$, and performing operations indicated, we have,

$U_{20}=28,383,163,779,300$, and $S_{20}=35,478,954,491,110$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The numbers in the problem may be represented under the following form :

1	6	35	180	921	4626
$5\times0+1$,	$5\times1+1$,	$5\times6+5$,	$5\times35+5$,	$5\times180+21$,	$5\times921+21$,

23215	116160	
$5\times4626+85$,	$5\times23215+85$,	etc.

The general term of the numbers 1, 5, 21, 82, etc., is $\frac{1}{3}(4^x-1)$, as can be easily found by Finite Differences. Expressing the $(2x-1)$th term of the above series by $F(2x-1)$, we have, by Finite Differences, $F(2x-1)=C.5^{2x-1}+C_1.4^x+C_2$. Substituting for x successively 1, 2, 3, we have the three equations: $5C+4C_1+C_2=1$, $125C+16C_1+C_2=35$, $3125C+64C_1+C_2=921$, whence $C=25/84$, $C_1=-1/7$, $C_2=1/12$.

$$\therefore\ F(2x-1)=\frac{5^{2x+1}}{84}-\frac{4^x}{7}+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{I}).$$

To find $F(2x)$, multiply $F(2x-1)$ by 5 and add $\frac{1}{3}(4^x-1)$, thus,

$$F(2x)=\frac{5^{2x+2}}{84}-\tfrac{8}{21}.4x+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{II}).$$

By summing the geometrical series $5^3+5^5+\ldots\ldots\ 5^{2x-1}$, $5^4+5^6+\ldots\ldots\ 5^{2x+2}$, $4+4^2+4^3+\ldots\ldots 4^x$, we find

$$\sum F(2x-1)=\frac{5^{2x+3}}{2016}-\frac{4^{x+1}}{21}+\tfrac{1}{12}x+\tfrac{269}{2016},\ \text{and}$$

$$\sum F(2x)=\frac{5^{2x+4}}{2016}-\tfrac{8}{63}.4^{x+1}+\tfrac{1}{12}x+\tfrac{399}{2016}.$$

Consequently $\displaystyle\sum_{x=1}^{x=2n-1}(x)=\frac{5^{2n+2}}{336}-\tfrac{5}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{35}{144}\ \ldots\ldots\ldots\ldots\ (\mathrm{III})$;

$$\sum_{x=1}^{x=2n}(x)=\frac{5^{2n+3}}{336}-\tfrac{11}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{47}{144}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{IV}).$$

The formulae I and III are to be employed for an odd number of terms, and II and IV for an even one. Thus, $F(20)=\dfrac{5^{22}}{84}-\tfrac{8}{21}.4^{10}+\tfrac{1}{12}=2838316377930\,0$;

$$\sum F(20)=\frac{5^{23}}{336}-\tfrac{11}{63}.4^{11}+\tfrac{10}{6}+\tfrac{47}{144}=35478954491110.$$

III. **Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.**

1. Write out to "n" terms the series : 1, 5, 25, 125, 625, ...5^{n+1}.
2. Begin at 3rd term and write the series : 4, 20, 100,.......4.5^{n-3}.
3. Begin at 5th term and write the series : 16, $4^2.5$, $4^2.5^{n-5}$.

....

4. Begin at $(n-1)$th term and write the series : $4^{\frac{n-1}{2}}$, $4^{\frac{n-2}{2}}.5$, n being even.
5. Begin at nth term and write the series : $4^{\frac{n-1}{2}}$, n being odd.

The nth term in the required series is the sum of all the numbers in the nth term of the above arrangement plus all that precede it. Denote the sum by s, then, if n is even,

$$s=\left\{\begin{array}{l} 1+5+25........+5^{n-1} \\ +4+4.5 ......+4.5^{n-3} \\ .... \\ + \qquad 4^{\frac{n-2}{2}}+4^{\frac{n-1}{2}}.5 \end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}........\frac{4^{\frac{n-2}{2}}(5^2-1)}{4}\ (1).$$

If n is odd,

$$s=\left\{\begin{array}{l} 1+5+35..........5^{n-1} \\ 4+4.5 ....4.5^{n-3} \\ .... \\ + \qquad 4^{\frac{n-1}{2}} \end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}........4^{\frac{n-1}{2}}\frac{(5-1)}{4}\quad (2).$$

(1) finally becomes : $\frac{1}{84}\{5^{n+2}+7-8.4^{\frac{n+2}{2}}\}$....................... ......(3),

and (2) becomes $\frac{1}{84}\{5^{n+2}+7-12.4^{\frac{n+1}{2}}\}$.................................(4).

(3) gives the even terms; (4) gives the odd terms. To sum the series :—
1st term by (4)$=\frac{1}{84}(5^3+7-12.4)$
2nd term by (3)$=\frac{1}{84}(5^4+7-8.4^2)$
3rd term by (4)$=\frac{1}{84}(5^5+7-12.4^2)$
4th term by (3)$=\frac{1}{84}(5^6+7-8.4^3)$
5th term by (4)$=\frac{1}{84}(5^7+7-12.4^3)$
....

nth term by (3)$=\frac{1}{84}(5^{n+2}+7-8.4^{\frac{n+2}{2}})$, n being even,

nth term by (4)$=\frac{1}{84}(5^{n+2}+7-12.4^{\frac{n+1}{2}})$, n being odd.

Denote the sum by S, then, *n being even*,

$$S = \tfrac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-11\frac{(4^{\frac{n+4}{2}}-16)}{3}\right\} \quad \ldots\ldots(5).$$

Similarly, n being odd,

$$S = \tfrac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-\frac{20.4^{\frac{n+3}{2}}-176}{3}\right\} \quad \ldots\ldots(6).$$

By (3), the 20th term is: $\frac{1}{84}\{5^{22}+7-8.4^{11}\}=28,383,163,779,300$.

By (5), the twenty terms are: $\frac{1}{84}\{\frac{5^{23}-125}{4}+140-11\frac{(4^{12}-16)}{3}\}$

$=35,478,954,491,110$.

IV. Solution by the PROPOSER.

Let it be required to sum to n terms and find the nth term of the series:

$1+6x+35x^2+180x^3+921x^4+4626x^5+23215x^6+116160x^7+\ldots\ldots$

Let the scale of relation be denoted by m, n, p, q.

$\therefore 921x^4=180qx^3+35px^2+6nx+m \ldots\ldots(1).$

$4626x^5=921qx^4\times 180px^3+35nx^2+6mx \ldots\ldots(2).$

$23215x^6=4626qx^5+921px^4+180nx^3+35mx^2 \ldots\ldots(3).$

$116160x^7=23215qx^6+4626px^5+921nx^4+180mx^3 \ldots\ldots(4).$

$\therefore m=20x^4,\ n=-24x^3,\ p=-x^2,\ q=6x.$

Since the series has a quadruple scale of relation it must be composed of the sum of four geometrical series. The ratios of these series will be the roots of the biquadratic equation

$r^4=6xr^3-x^2r^2-24x^3r+20x^4 \ldots\ldots(5).$

$\therefore r_1=2x,\ r_2=-2x,\ r_3=5x,\ r_4=x.$

Let a_1, a_2, a_3, a_4 be the first terms of these sets of series; then

$a_1+a_2+a_3+a_4=1 \ldots\ldots(6).$

$a_1r_1+a_2r_2+a_3r_3+a_4r_4=2a_1-2a_2+5a_3+a_4=6 \ldots\ldots(7).$

$a_1r_1^2+a_2r_2^2+a_3r_3^2+a_4r_4^2=4a_1+4a_2+25a_3+a_4=35 \ldots\ldots(8).$

$a_1r_1^3+a_2r_2^3+a_3r_3^3+a_4r_4^3=8a_1-8a_2+125a_3+a_4=180 \ldots\ldots(9).$

$\therefore a_1=-2/3,\ a_2=2/21,\ a_3=125/84,\ a_4=1/12.$

Hence the series are:

$-2/3-4x/3-8x^2/3-16x^3/3-32x^4/3-64x^5/3-\ldots\ldots(10).$

$2/21-4x/21+8x^2/21-16x^3/21+32x^4/21-64x^5/21+\ldots\ldots(11).$

$125/84+625x/84+3125x^2/84+15625x^3/84$

$+78125x^4/84+390625x^5/84+\ldots\ldots(12).$

$$1/12+x/12+x^2/12+x^3/12+x^4/12+x^5/12+\ldots\ldots\ldots\ldots\ldots(13).$$

Let $A_n{}^1$, $A_n{}^2$, $A_n{}^3$, $A_n{}^4$, $S_n{}^1$, $S_n{}^2$, $S_n{}^3$, $S_n{}^4$ represent the nth terms; and the sum of n terms of the series (10), (11), (12), (13). Then,

$$A_n{}^1=-\tfrac{2}{3}(2x)^{n-1},\ A_n{}^2=\tfrac{2}{21}(\pm 2x)^{n-1},\ A_n{}^3=\tfrac{125}{84}(5x)^{n-1},\ A_n{}^4=\tfrac{1}{12}x^{n-1},$$

$$S_n{}^1=-\tfrac{2}{3}\left(\frac{2^nx^n-1}{2x-1}\right),\ S_n{}^2=\tfrac{2}{21}\left(\frac{\pm 2^nx^n-1}{-2x-1}\right),\ S_n{}^3=\tfrac{125}{84}\left(\frac{5^nx^n-1}{5x-1}\right),$$

$$S_n{}^4=\tfrac{1}{12}\left(\frac{x^n-1}{x-1}\right).$$

Let A_n, S_n, be the nth term and the sum of n terms of the original series.

$$\therefore\ A_n=\tfrac{1}{84}\{5^{n+2}+7(1-2^{n+2})\mp 2^{n+2}\}x^{n-1}.$$

$$S_n=\tfrac{1}{84}\left\{\frac{125(5^nx^n-1)}{5x-1}-\frac{56(2^nx^n-1)}{2x-1}+\frac{8(\pm 2^nx^n-1}{-2x-1}+\frac{7(x^n-1)}{x-1}\right\}.$$

The upper sign to be used when n is even. Now let $x=1$, $n=20$, and we will get the required results for the problem. $A_{20}=28383163779300$, the number the twentieth year; $S_{20}=35478954491110$, the number in twenty years.

Also solved by *EDWARD R. ROBBINS*.

65. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at the same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained 5-8 as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was 43 9-20 per cent. How much land did each buy, and what did each receive per acre at the sale?

I. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

Let x, y, and z be the number of acres bought by A, B, and C, respectively. $\therefore\ x+y+z=200\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

Since the selling price is \$286.90 and the gain per cent. is 43.45, the cost is \$200. Let m=cost per acre; then mx, my, and mz represent the cost of the shares of A, B, and C, respectively. $\therefore\ m(x+y+z)=200$. $\therefore\ m=1$. $\therefore$ the cost of the share of each=number of acres he bought.

x=A's gain per cent., and $5x/8$=B's gain per cent.

$\therefore\ x+x^2/100+y+5xy/800+z-\$9.10=\$286.90.$

$\therefore\ x^2/100+5xy/800=\$96.$ $\therefore\ 8x^2+5xy=76800.$

$$\therefore\ y=\frac{76800-8x^2}{5x}=\frac{15360}{x}-\frac{8x}{5}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If the number of acres bought by each is to be integral, then (1) and (2) are to be solved for *positive integral* values of x, y, and z. Since y is to be integral, x must be a factor 15360 and must be divisible by 5. $15360=5\times3\times2^{10}$. $\therefore$ the factors of 15360 which are divisible by 5, are 5, 10, 15, 20, 30, 40, 60, 80, 120, 160, etc. If x has any of these values less than 80, z will be negative; if x has any values greater than 80, y is negative. If $x=80$, $y=64$, and $z=56$. $\therefore$ 80, 64, and 56 are the shares of A, B, and C.

The amounts each received per acre at the sale are easily found to be \$1.80, \$1.50, and \$0.83$\frac{3}{4}$.

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics in the Lawrenceville School, Lawrenceville, New Jersey.

Let x, y, and $200-x-y$ represent the number of acres which A, B, and C bought, respectively. Then by the problem,

$$x+x^2/100+y+5xy/800+200-x-y-9.10=286.90.$$

This gives $8x^2+5xy=76,800$; or $y=(76800-8x^2)/5x$. Solving for positive integers in x, we have, when

$$x=75,\quad 80,\ 85,\quad 90,$$
$$y=107\tfrac{3}{7},\ 64,\ 44\tfrac{12}{17},\ 26\tfrac{2}{3},$$

Accepting the integral values we obtain:
A's purchase consisted of 80 acres and sold for \$144;
B's purchase consisted of 64 acres and sold for \$96;
C's purchase consisted of 56 acres and sold for \$46.90.
Hence A received \$1$\frac{4}{5}$ per acre; B, \$1$\frac{1}{2}$; and C, \$$\frac{67}{80}$.

III. Solution by H. C. WILKES, Skull Run, West Virginia.

Since by the terms of the problem the price paid for the land was \$1 per acre, let $8x$, y, z be the number of acres bought, and the number of dollars paid, by A, B, and C, respectively.

Then $8x+y+z=200\ldots(1)$. $8x+64x^2/100+y+5xy/100+z=296\ldots(2)$.

Subtracting (1) from (2), and clearing, $64x^2+5xy=9600$. Factoring, $x(64x+5y)=10(960)$. Let $x=10$; then $5y=320$, and $y=64$.

$\therefore$ 80, 64, 56 are numbers satisfying the conditions. See solution of a similar problem on page 76 of Vol. II.

IV. Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.

Let x, y, and z represent the shares of A, B, and C, respectively. $x+y+z=200\ \ldots\ldots(1)$. Since C lost \$9.10, he must have bought at least 9.10 acres. Therefore 190.90 is the maximum limit of $x+y$.

$$x+x^2/100+y+xy/160+z=296\ldots\ldots(2).$$

(1) in (2) gives $x^2/100+xy/160=96$. $\therefore y=(76800-8x^2)/5x\ldots\ldots(3)$.

$\therefore$ $190.90>x+(76800-8x^2)/5x$. $\therefore$ $190.90>(76800-3x^2)/5x$.
As x decreases, $(76800-3x^2)/5x$ increases.

$\therefore$ the equation $(76800-3x^2)/5x=190.90$ .... (4)

gives the minimum limit of x.

$\therefore$ $66.48+$ is the minimum limit of x .... (5).

From (3), $y=(76800-8x^2)/5x$, we get, since y must have some value, $76800>8x^2$; hence $8x^2=76800$ gives maximum limit of x. $\therefore$ $97.97+$ is the maximum limit of x. Hence, any values of x between $66.48+$ and $97.97+$ will satisfy the conditions of the problem. *Example:* Let $x=77\frac{1}{7}$. Then from (3) $y=75\frac{43}{63}$; $\therefore$ $z=47\frac{11}{63}$.

$\therefore$ A received \$$136.65\frac{136}{441}$; B received \$$112.17\frac{103}{441}$. $\therefore$ C received \$$38.07\frac{203}{441}$; but he paid \$$47.17\frac{203}{441}$. $\therefore$ C lost \$9.10.

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Maryland.

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4\times\sqrt{6-\sqrt{2}}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when B's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when C's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when D's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when E's worth is added to e times what A, B, C, and D are worth, it is equal to r.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

51. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustum.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A complete solution of this problem without any assumptions would be a task greater than I care to undertake at present. We will, therefore, assume the cone to be one of revolution. Let $2h$=height of frustum, R, r radii of the lower and upper bases, respectively, l, m, p the coordinates of the vertex.

$x^2/a^2+y^2/b^2+z^2/c^2=1$, the equation to the ellipsoid.

$\therefore$ $(p-z)^2+(n-y)^2=[(R-r)/2h]^2(m-x)^2$ is the equation to the cone.

We will further assume that this cone is the tangent cone to the maximum ellipsoid, then the equation to the cone is

$$(m^2/a^2+n^2/b^2+p^2/c^2-1)(x^2/a^2+y^2/b^2+z^2/c^2-1)$$
$$=(mx/a^2+ny/b^2+pz/c^2-1)^2.$$

From these two equations to the cone we get $n=p=0$.

$[(R-r)/2h]^2=Rr/(m^2-h^2)$ or $m=[(R+r)/(R-r)]h$.

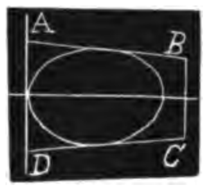
$\therefore$ The center of the frustum and the center of the ellipsoid coincide, and the ellipsoid is one of revolution.

$\therefore$ $x^2/a^2+(y^2+z^2)/b^2=1$ is its equation. $\therefore$ $a=h$, $b=\sqrt{Rr}$.

$V=\frac{4}{3}\pi hRr$=volume of maximum ellipsoid.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

The figure shows vertical section of frustum and inscribed ellipsoid, with axis of x coinciding with axis of cone, and axis of y in base. Let d and c be radii of bases, and h the altitude. Then $(0, d)$ and (h, c) represent points A and B, respectively.



$(x-a)^2/a^2+y^2/b^2=1$ is equation to inscribed ellipse. $\therefore$ equation to AB, as tangent to ellipse, is

$$a^2yy_1+b^2(x-a)(x_1-a)=a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

x_1 and y_1 being coordinates of point of contact. Substituting coordinates of A and B for x and y in (1),

$$\left.\begin{aligned} a^2dy_1+b^2(-x)(x_1-a)=a^2b^2 \\ a^2cy_1+b^2(h-a)(x_1-a)=a^2b^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solving (2) for x_1 and y_1,

$$\left.\begin{aligned} x_1=ahd/(bd-ad+ac) \\ y_1=b^2h/(bd-ad+ac) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Substituting from (3) for x and y in equation of ellipse and solving we obtain $b^2=[(hd^2+2ad(c-d)]/h$.

Now volume of ellipsoid $V=4/3(\pi ab^2)=4\pi/3h[ad^2h+2a^2d(c-d)]$.

$$dV/da=4\pi/3h[d^2h+4ad(c-d)]\ldots\ldots\ldots\ldots\ldots(4).$$

Equating (4) to 0, we find $a=dh/4(d-c)$..............................(5),

$$\text{and } b^2=d^2/2\ldots\ldots\ldots\ldots\ldots(6).$$

Also, $d^2V/da^2=16\pi d(c-d)/3h$, which is negative since $d>c$. Now the ellipsoid will be entirely within the frustum if $2a$ is not greater than h, which from (5) gives, $dh/2(d-c)$ is not greater than h or c is not greater than $\frac{1}{2}d$. So volume of maximum ellipsoid=

$\frac{4\pi}{3}\cdot\frac{dh}{4(d-c)}\cdot\frac{d^2}{2}=\frac{\pi}{6}\frac{d^3h}{d-c}$, if c is not greater than $\frac{1}{2}d$, or $\frac{4\pi}{3}hb^2$, if $c>\frac{1}{2}d$, the latter result being true, since (4) shows but one maximum, and V is a continuous function of A.

III. **Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

Take the base of the frustum as the plane xz, and the axis of the frustum as the axis of y. We may, without loss of generality, take one axis parallel to the axis of z. The equation of the ellipsoid may then be written:

$$Ax^2+By^2+Cxy+Dx+Ey+Hz^2+F=0\ldots\ldots\ldots\ldots\ldots(1).$$

We find the axes of the ellipsoid to be:

$$a=\sqrt{R/P},\ b=\sqrt{R/Q},\ c=\sqrt{R/H},$$

where $R=F(c^2-4AB)+AE^2+BD^2-CD^2)/(4AB-C^2)$.

$P=1/2[A+B\pm\sqrt{(A-B)^2+C^2}]$,

$Q=1/2[A+B\mp\sqrt{(A-B)^2+C^2}]$,

Volume of ellipsoid$=4/3(\pi abc)$

$$=\tfrac{8}{3}\pi\frac{[F(C^2-4AB)+AE^2+BD^2-CDE]^{\frac{3}{2}}}{[4AB-C^2]^2}\cdot\frac{1}{\sqrt{H}}\ldots\ldots\ldots\ldots(2).$$

A little consideration will show that the ellipsoid to be a maximum *must touch* the larger base of the frustum and also the conical surface. The condition that it touch the lower base is $D^2-4AF=0$..............................(3).

The condition that it shall not cut the upper base is

$$(Ch+D)^2-4A(Bh^2+Eh+F(<0\ldots\ldots\ldots\ldots(4),$$

where h is the altitude of the frustum.

To find the condition that the ellipsoid shall be tangent to conical surface, we assume the equation of complete cone to be:

$$m^2(x^2+z^2)=(y-k)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

For intersection of (1) and (5),

$$(A-H)m^2x^2+(Bm^2+H)y^2+Cm^2xy+Dm^2x+(Em^2-2k)+(Hk^2+Fm^2)=0.$$

If this ellipse have no axes,

$$(Hk^2+Fm^2)[c^2m^2-4(A-H)(Bm^2+H)]+(A-H)(Em^2-2k)^2+(Bm^2+H)D^2m^2$$
$$-CD(Em^2-2k)m^2=0.$$

Solving this for B we obtain,

$$B=\frac{CD(Em^2-2k)m^2-(A-H)(Em^2-2k)^2-C^2m^2(Hk^2+Fm^2)}{m^2[D^2m^2-4(A-H)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

Substitute the value of A given in (3),

$$B=\frac{4FCD(Em^2-2k)m^2-(D^2-4FH)(Em^2-2k)^2-4FC^2m^2(Hk^2+Fm^2)}{m^2[FD^2m^2-(D^2-4FH)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

If we were then to substitute these values of A and B in equation (2), we should obtain a value of V which contains the variables C, D, E, F, and H, independent, except as to the condition given in (4). By the ordinary methods of maximum and minimum, five equations can be formed and the maximum critical values of the five letters determined. But life is too short to do this.

If we assume that two of the axes are parallel to the bases of the frustum, we obtain $V=\frac{4}{3}\pi\tan^2\phi(h^2b-2hb^2)$, where h=altitude of complete cone, ϕ=semi-angle of cone, and b=semi-vertical axis of ellipsoid. From this for maximum, $b=p/4$.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area?

I. Solution by the PROPOSER.

If we take the point where the light with intensity l is situated as the origin of coordinates, we have readily from the principles of Optics,
$I=ly/(x^2+y^2)^{\frac{3}{2}}+my/[(a-x)^2+y^2]^{\frac{3}{2}}$, x and y being the coordinates of the bull's-eye.

$$\frac{dI}{dx}=\frac{-3lxy}{(x^2+y^2)^{\frac{5}{2}}}+\frac{3m(a-x)y}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0 \;\ldots\ldots\ldots\ldots\ldots (1).$$

$$\frac{dI}{dy}=\frac{l(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}=\frac{m[(a-x)^2-2y^2]}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0 \;\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) $y=0$ .... (a),

$$\text{or } \frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=\frac{m}{l}\cdot\frac{a-x}{x} \;\ldots\ldots\ldots\ldots\ldots (b).$$

$$\text{From (2) } \frac{[(a-y)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=-\frac{m}{l}\cdot\frac{(a-x)^2-2y^2}{x^2-2y^2} \;\ldots\ldots\ldots\ldots\ldots (c).$$

From (b) and (c) $y^2=x(a-x)/2$ .... (d).

$$\text{From } (b) \;\; y=\pm x^{\frac{1}{5}}(a-x)^{\frac{1}{5}}\sqrt{\frac{m^{\frac{2}{5}}x^{\frac{8}{5}}-l^{\frac{2}{5}}(a-x)^{\frac{8}{5}}}{l^{\frac{2}{5}}x^{\frac{2}{5}}-m^{\frac{2}{5}}(a-x)^{\frac{2}{5}}}} \;\ldots\ldots\ldots\ldots\ldots (e).$$

By (a) and (d), $x=0$; $x=a$; that is, the lights themselves must be used as bull's-eye. By (a) and (e) we obtain the additional condition $x=al^{\frac{1}{4}}/(l^{\frac{1}{4}}+m^{\frac{1}{4}})$, which is the point of minimum illumination on the line joining the two lights. Other critical points will be obtained by solving (d) and (e) simultaneously,—a task which seems to be almost impossible.

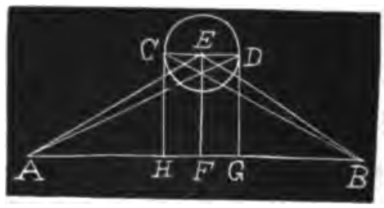
II. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let A, B, be the lights, intensities m, n; E the center of the target, radius $ED=r$, $AB=a$, $AF=x$, $EF=z$, $\angle DAB=\theta$, $\angle CBA=\phi$.

$\therefore\; m\sin\theta/AD^2+n\sin\phi/BC^2=I$.

$\sin\theta=z/AD=z/\sqrt{z^2+(x+r)^2}$,

$\sin\phi=z/\sqrt{z^2+(a-x+r)^2}$.



$$\therefore\; \frac{mz}{\{z^2+(x+r)^2\}^{\frac{3}{2}}}+\frac{nz}{\{z^2+(a-x+r)^2\}^{\frac{3}{2}}}=I.$$

Differentiating with reference to z,

$$\frac{m(x+r)^2-2mz^2}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}+\frac{n(a-x+r)^2-2nz^2}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}}=0 \;\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to x,

$$\frac{m(x+r)}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}=\frac{n(a-x+r)}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}} \;\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), $z=\sqrt{\tfrac{1}{2}(x+r)(a-x+r)}$. This value of z in (2) gives

$$\frac{m(x+r)}{\{(x+r)(a+x+3r)\}^{\frac{3}{2}}}=\frac{n(a-x+r)}{\{(a-x+r)(2a+3r-x)\}^{\frac{3}{2}}},$$

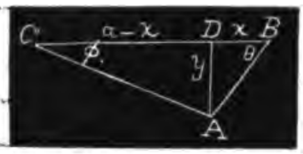
an equation of the eighth degree to find x.

If $m=n$, $x=\frac{1}{2}a$, $z=(\frac{1}{2}a+r)\sqrt{\frac{1}{2}}=\frac{1}{2}(\frac{1}{2}a+r)\sqrt{2}$.

If $n=0$, $x=0$, $z=\frac{1}{2}r\sqrt{2}$.

III. **Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.**

Let A be any position of target, $AD(=y)$ be perpendicular from A to BC, the line connecting the positions of the two lights. Let x equal part of $BC(=a)$ cut off by AD. By laws of light, intensity of light received from B at A



$$=\frac{m\sin\theta}{AB^2}=\frac{m}{x^2+y^2}\times\frac{y}{\sqrt{x^2+y^2}}=\frac{my}{(x^2+y^2)^{\frac{3}{2}}}.$$

Similarly, that received from $C=\dfrac{ny}{[(a-x)^2+y^2]^{\frac{3}{2}}}$.

Then total intensity at A or $u=my(x^2+y^2)^{-\frac{3}{2}}+ny[(a-x)^2+y^2]^{-\frac{3}{2}}$. ..(1).

$$du/dx=-3mxy(x^2+y^2)^{-\frac{5}{2}}+3n(a-x)y[(a-x)^2+y^2]^{-\frac{5}{2}}\text{..........(2)}.$$

$$du/dy=m(x^2+y^2)^{-\frac{3}{2}}-3my^2(x^2+y^2)^{-\frac{5}{2}}+n[(a-x)^2+y^2]^{-\frac{3}{2}}-3ny^2[(a-x)^2+y^2]^{-\frac{5}{2}}\text{..........(3)}.$$

Equating (3) to 0, we have

$$y=0\text{..........(4)}.$$

$$\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n(a-x)}{mx}\text{..........(5)}.$$

Equating (3) to 0, we have

$$[(a-x)^2+y^2]^{-\frac{5}{2}}\{3ny^2-n[(a-x)^2+y^2]\}=(x^2+y^2)^{-\frac{5}{2}}[m(x^2+y^2)-3my^2],$$

$$\text{or }\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}\text{..........(6)}.$$

Solving (4) and (6), $\dfrac{(a-x)^5}{x^5}=\dfrac{-n(a-x)^2}{mx^2}$, which gives

$$\left\{\begin{array}{l} x=a \text{ or } 0 \text{ or } \dfrac{a}{1-\sqrt[3]{m\div n}} \\ \text{and } y=0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From (5) and (6) $\dfrac{n(a-x)}{mx}=\dfrac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}$, and $y^2=\dfrac{x(a-x)}{2}$......(8)

Instead of finding second differential coefficients, substitute from (7) in (1), $x=a$ and $x=0$, make $n=\infty$. $x=\dfrac{a}{1-\sqrt[3]{n\div m}}$, makes $n=0$.

We can show that (8) does not produce any new condition for a maximum. To make y real x is not <0 nor $>a$.

If $x=a$ or 0, we have the values found in (7). Now for any value of x between 0 and a, y in (8) is seen to be finite, and n in (1) is also finite.

So $x=a$ or $x=0$ with $y=0$, producing the only infinite values of n indicate the positions of maximum intensity of illumination to be directly in front of either light.

PROBLEMS.

59. Proposed by MOSES C. STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Indiana.

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[*From Forsyth's Differential Equations.*]

60. Proposed by SETH PRATT, C. E., Assyria, Michigan.

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, whose axis is the axis of a sphere, and whose vertex is at the surface of the sphere. Required the height of the cone and the diameter of its base.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26.

After carefully reading Dr. Martin's "Reply to Replies on Problem 26," we see no reason for changing our opinion respecting the solution we have been defending. We may, however, be led to agree with Dr. E. H. Moore, Dr. William

Hoover, and Prof. Henry Heaton, that there is no *correct* solution of the problem. That is to say, in so much as the problem is stated in the indefinite form, a solution taking any one of the elements of a triangle of which the area is a function will lead to a correct result. But it does seem to us that this statement, while it is true in general, does not apply in this case. It was asked of us during the summer whether anyone had sent in a solution assuming the altitude as the variable. Now we think it is quite clear that a solution which assumes the altitude of the triangle as the variable can, in no way be correct, for the solution would include not only right triangles, but oblique triangles as well. The result is

$$2\int_0^{\frac{1}{2}a} \tfrac{1}{2}apdp \div \int_0^{\frac{1}{2}a} dp = \tfrac{1}{4}a^2, \text{ where } p \text{ is the altitude.}$$

But if p is made a function of the angle at the center of the circle subtended by a side, the result will be $\dfrac{a^2}{2\pi}$. We think, however, that this controversy has been carried on long enough, and therefore it is desirable that it close without further discussion. EDITOR.

NOTE ON PROBLEM 29.

BY HENRY HEATON.

When the points to which r is measured are distributed symmetrically with respect to the minor axis the different radii vectores may be arranged in pairs such that the sum of the lengths of each pair will be $2a$. Hence, using Dr. Hoover's notation, m'' and m''' each equal a. m''' may be shown to equal a by the calculus, thus:

$$r = a - ex,\ d\theta = \frac{(a^2 - e^2a^2)^{\frac{1}{2}}\,dx}{(a^2 - x^2)^{\frac{1}{2}}}.$$

Hence $$m''' = \int_{-a}^{+a} \frac{(a-ex)(a^2 - e^2x^2)^{\frac{1}{2}}\,dx}{(a^2 - x^2)^{\frac{1}{2}}} \div \int_{-a}^{+a} \frac{(a^2 - e^2x^2)^{\frac{1}{2}}\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= a\int_{-a}^{+a} \frac{(a^2 - e^2x^2)^{\frac{1}{2}}\,dx}{(a^2 - x^2)^{\frac{1}{2}}} \div \int_{-a}^{+a} \frac{(a^2 - e^2x^2)^{\frac{1}{2}}\,dx}{a^2 - x^2} = a.$$

A fourth very obvious case of this problem is when the distances are measured at equal intervals of time.

Then $$m'''' = \int rdA \div \int dA = \int_0^{\pi} \frac{r^3}{2}\,d\theta \div \int_0^{\pi} \frac{r^2}{2}d\theta$$

$$= \frac{a^2(1-e^2)^3}{b\pi}\int_0^{\pi} \frac{d\theta}{(1-e\cos\theta)^3} = \frac{a^2}{2b}\left(3(4+e^2)(1-e^2)^{\frac{1}{2}} - 10(1-e^2)^2\right).$$

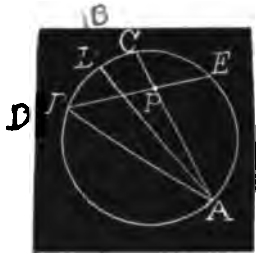
Corollary. Let $e=0$, then $m''''=a$, as it evidently should.

32. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of the random sector whose vertex is a random point in a given circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point. Through P draw the chords AC, DE forming the sector DPC. From A draw the diameter AB and the chord AD. Let $AB=2r$, $AP=z$, $\angle BAP=\varphi$, $\angle BAD=\theta$, area $DPC=u$, $\Delta=$required average. Then



$$u=r^2(\varphi-\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{2}\sin 2\varphi)-rz\cos\theta\sin(\varphi-\theta).$$

The limits of θ are $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of φ, θ and $\frac{1}{2}\pi$; of z, 0 and $2r\cos\varphi=z'$.

$$\therefore\ \Delta=\frac{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_0^{z'}u\,d\theta\,d\varphi\,z\,dz}{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_0^{z'}d\theta\,d\varphi\,z\,dz}=\frac{2}{\pi^2r^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_0^{z'}u\,d\theta\,d\varphi\,z\,dz,$$

$$=\frac{2r^2}{3\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}[3\cos^2\varphi(2\varphi-2\theta-\sin 2\theta+\sin 2\varphi)-8\cos\theta\cos^3\varphi\sin(\varphi-\theta)]d\theta\,d\varphi$$

$$=\frac{r^2}{12\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}(3\pi^2-12\pi\theta+\theta^2-16\cos^2\theta+4\sin^2\theta\cos^2\theta)d\theta=\frac{37\pi r^2}{144}-\frac{5r^2}{8\pi}.$$

Also solved by the *PROPOSER*.

33. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all regular polygons having a constant apothem.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $a=$constant apothem, $2x=$side, $2\theta=$central angle of polygon.

$\therefore\ \frac{\pi}{\theta}=$number of sides, $\frac{\pi ax}{\theta}=$area of polygon.

$$\therefore\ \Delta=\text{average area}=\pi a\frac{\int_0^{a\sqrt{3}}\frac{x\,dx}{\theta}}{\int_0^{a\sqrt{3}}dx}=\frac{\pi}{\sqrt{3}}\int_0^{a\sqrt{3}}\frac{x\,dx}{\theta}$$

$$=\frac{\pi a^2}{\sqrt{3}}\int_0^{\frac{1}{3}\pi}\frac{\tan\theta\sec^2\theta\,d\theta}{\theta}\quad\text{where } x=a\tan\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}\left(\frac{\tan\theta}{\theta}\right)^2 d\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}(1+\tfrac{2}{3}\theta^2+\tfrac{17}{45}\theta^4+\tfrac{62}{315}\theta^6+\ldots\ldots)d\theta,$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi^2 a^2}{6\sqrt{3}}\left(1+\frac{2\pi^2}{81}+\frac{17\pi^4}{18225}+\frac{62\pi^6}{1607445}+\ldots\ldots\right)=3.8693a^2 \text{ nearly.}$$

Also solved by the ***PROPOSER***.

34. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two points are taken at random on the circumference of a semicircle. Find the chance that their ordinates fall on either side of a point taken at randon on the diameter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point on the diameter AE. Draw BP perpendicular to AE. Then one point must fall somewhere, as at C, on arc AB, the other somewhere, as at D, on arc BE. The chance thus obtained must be doubled as D might fall on AB and C on BE.

Let AO=unity, $\angle BOA=\theta$, $\angle COA=\varphi$, $\angle DOA=\psi$.

Then $OP=\cos\theta$. $\therefore d(OP)=-\sin\theta d\theta$.

Let p=required chance.

$$\text{Then } p=\frac{\int_0^{\pi}\int_0^{\theta}\int_{\theta}^{\pi}\sin\theta d\theta d\varphi d\psi}{\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\sin\theta d\theta d\varphi d\psi}=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\theta}\int_{\theta}^{\pi}\sin\theta d\theta d\varphi d\psi$$

$$=\frac{1}{\pi^2}\int_0^{\pi}(\pi\theta-\theta^2)\sin\theta d\theta=\frac{4}{\pi^2}.$$

PROBLEMS.

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42. Proposed by CHARLES E. MYERS, Canton, Ohio.

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords,.(1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

EDITORIALS.

Prof. John N. Lyle, of Westminster College, has resigned his position on account of ill health, and is now living in Bentonville, Arkansas.

We shall be pleased to have our subscribers send us the names of persons likely to subscribe for the MONTHLY, in order that we may send such persons sample copies.

Any reader of the MONTHLY having a copy of *Salmon's Higher Plane Curves*, third edition, and wishing to sell the same, should write to us stating the price of the book.

We have only six complete sets of Volumes I and II, of the MONTHLY. Volume I will be sent to any address in the United States or Canada for $2.00; Volume II will be sent on receipt of $2.50.

Prof. Robert J. Aley, of Indiana University, is now studying mathematics in the University of Pennsylvania, having received a Mathematical Fellowship in that Institution last spring.

In our August-September number, we sent out bills to all those who are owing us. We hope that the matter of remittance may receive the attention of all those who are in arrears, as the MONTHLY is greatly in need of funds. All bills not paid by December 31st will be sent to an attorney for collection.

Prof. A. B. Nelson, of Centre College, Kentucky, says, in a letter of October 13th, "You deserve the thanks of mathematicians in this country for your self-sacrificing labors in behalf of our favorite science." We desire to thank Professor Nelson as well as many others who have thus expressed their appreciation of our labor. Surely it is a labor of love.

BOOKS AND PERIODICALS.

Elementary Solid Geometry and Mensuration. By Henry Dallas Thompson, D. Sc., Ph. D., Professor of Mathematics in Princeton University. 8vo. Cloth, 200 pages. Price, $1.25. New York: The Macmillan Co.

In this book, the author lays the foundation of his subject in clear cut and accurate *definitions* and well illustrated *postulates*. The diagrams are very fine, showing very accurately to the eye the relation of the points, lines, and planes. There are numerous original exercises scattered throughout the book. B. F. F.

The Elements of Algebra. Adapted for use in High Schools, Academies, and Colleges. By Lyman Hall, Graduate United States Military Academy, and Professor of Mathematics, Georgia School of Technology. 8vo. Cloth and Leather Back, 368 pages. Chicago: American Book Co.

This work is intended for beginners who have mastered the principles of any good common school Arithmetic. The familiar methods of arithmetic are preserved, in order to gradually convince the student that algebra is merely an extension of the mathematical knowledge he already possesses. *Preface.* B. F. F.

Trigonometry for Beginners. By the Rev. J. B. Lock, M. A., Fellow of Gonville and Caius College, Cambridge, Formerly Master at Eaton. Revised and Enlarged for the use of American Schools, by John A. Miller, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Professor (elect) of Mechanics and Mathematical Astronomy, Indiana University. Large 8vo. Cloth, 148 and 64 pages. Price, $1.10.

As we have not seen the original book, we do not know just how materially Professor Miller has changed it. He tells us in his Preface that it differs from the original, chiefly in the following particulars: (1) The subject matter of Chapter VII formerly followed that of Chapters VIII and IX; (2) the addition formulæ are proved for angles of any magnitude, and for more than two angles; (3) a chapter on Inverse Trigonometric Functions; and two chapters on Spherical Trigonometry have been added; (4) logarithmic and trigonometric tables have been inserted. Some of the trigonometrical formulæ are very neatly established by Geometrical Proof. B. F. F.

A School Algebra. Designed for use in High Schools and Academies. By Emerson E. White, A. M., LL. D., Author of "Series of Mathematics," "Elements of Pedagogy," "School Management," etc. 8vo. Cloth and Leather Back, 394 pages. Chicago: American Book Co.

The author's aim has been to prepare a school algebra that is pedagogically sound as well as mathematically accurate. Few educators will question Dr. White's ability to write a work pedagogically sound, but many mathematicians, upon examination of his treatment of *Undetermined Coefficients*, Chapter XXI., will question the mathematical accuracy of his text on algebra. His treatment of Undetermined Coefficients is that given in most algebras written during the last and present century. This demonstration is now pretty generally admitted to be incorrect, and correct demonstrations are being published in most recent works. However, upon the whole, the book is one well suited for the purpose for which it is written. B. F. F.

Elements of Geometry. By Andrew W. Phillips, Ph. D., and Irving Fisher, Ph. D., Professors in Yale University. Large 8vo. Cloth and Leather Back, 540 pages. Price, $1.75. New York: Harper & Bros.

There are several features in this work that make it especially interesting. Of these the most prominent are the beautiful diagrams. These are photo-engravings arranged side by side with skeleton drawings of geometrical figures. The photographs were taken from actual models recently constructed for use in the class-rooms of Yale University. In this respect the work excels anything that has yet appeared in this country. The work is characterized by clearness of presentation, both in the form of the diagrams and the natural and symmetrical methods of proof. The book closes with a short but very clear treatment of Modern Geometry. This will be helpful to those teachers who desire a knowledge of

the three kinds of Geometries. We believe that this work is destined to be very extensively used throughout the country. B. F. F.

A History of Elementary Mathematics, with Hints and Methods of Teaching. By Florian Cajori, Ph. D., Professor of Physics in Colorado College. 8vo. Cloth, 304 pages. Price, $1.50. New York: The Macmillan Co.

The book is by no means an abridged edition of the author's *History of Mathematics*. It is an entirely new book giving a somewhat detailed account of the rise, struggle, and progress of Arithmetic, Algebra, and Geometry. The book should be read by all teachers of these subjects, and by mathematical students generally. B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single numbes, 10 cents. Irvington-on-the-Hudson, New York.

The October number contains the following: A Summer Tour in the Scottish Highlands; The Story of a Child Trainer; The Perils and Wonders of a True Desert; A Modern Fairy Tale; Hofman's Object Lesson; Personal Recollections of the Tai-Ping Rebellion; The Modern Woman Out-of-Doors; The True History of our Cooks; To a Hyacinth Bulb (poem). B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

In the September and October numbers of *The Review of Reviews*, the editor has given a remarkably fair and unprejudiced account of the progress of the present political campaign. It is a great satisfaction, after having read statements in the daily papers, which are believed to be misrepresenting, to go to *The Review of Reviews* and get the facts there given by its able editor. The November number contains a very able article on the "Summing Up of the Vital Issues of 1896," by Rev. Dr. Lyman Abbott. Also the question "Would Free Coinage Benefit Wage Earners?" is debated by Dr. Chas. B. Spahr and Prof. Richmond Mayo-Smith. This number also offers a remarkable symposium of current thought on "What Should be Done with Turkey?" The MONTHLY suggests in answer to this question, that Turkey be given a material and substantial roast by the civilized world.

B. F. F.

ERRATA.

After the word, ellipses, page 181, problem 60, insert, "passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices."

Page 205, line 1, for "$\frac{2}{3}$" read $\frac{3}{2}$.

Page 205, line 1, for "$\frac{3}{2}$" read $\frac{2}{3}$.

Page 205, line 12, for "$4a^3$" read $2a^3$.

Page 206, line 3, for "$(5x^{30}$" read $(5x)^{30}$.

Page 217, line 15, for "y" read z.

Throughout the solution to problem 34, Mechanics, for "E_o^π" read F_o^π.

// THE
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NUMBER AND FRACTIONS.

By J. K. ELLWOOD, A. M., Pittsburg, Pennsylvania.

A clear understanding of what *number* is and what gives rise to the number idea removes all difficulty from the grasping of the *fraction* idea.

Number does not inhere in objects, cannot be perceived by the senses; otherwise the mere presentation of 2, 3......n objects to the senses would give rise to the idea of number. There is in every sound mind a *measuring* instinct, which, in the nature of things, is just as essential to life and progress as is memory. Both the physical and ideal worlds are full of entities—vague wholes—which the mind must *measure* for the purpose of making them more definite. Measuring requires a "unit of measure." Naturally the first measurements made by a child are vague; as when he measures (counts) the chairs in a room, the marbles in his pocket, the fingers on his hand. His units of measure—chair, marble, finger—are indefinite, as are the results of his processes. A later stage involves *exact* measurements; i. e., an exactly defined unit of measure is used. A whole (of quantity), say a piece of cloth, is to be measured—made definite in value. A *yard* (exactly defined as 3 feet or 36 inches) is taken as the unit and applied (say) *ten* times. Then *ten* repetitions of the unit is the *number*. Considered by itself the *ten* is *pure number*, the result of a purely mental process; it expresses the *ratio* of the measured *quantity* to the measuring unit. Applied to the unit of measure, then ten expresses the numerical value of the measured quantity—10 yards of cloth. This ten yards, it is evident, is *quantity*, not number. It is what arithmetics erroneously call "concrete number." In this example the pure number indicates either of two things: (a) that the unit is taken *ten times*, or (b) that ten parts (units) are taken *one time*. It answers the question "how many?" Applied to the unit, it answers the question "how much?"

The number and unit of measure *together* give the absolute magnitude of the quantity; the number *alone* gives the relative value. Hence we may say that *number is the ratio of the quantity measured to the unit of measure.*

It is plain that *any* quantity may be used as a unit of measure. Measurement is more exact when this unit is itself made up of a definite number of equal parts—measured by some other unit, which may be called "primary" to distinguish it from the actual or direct unit of measure, which may be called "derived." Thus, if the unit of measure is three feet and it is taken ten times, we have the primary unit *one foot*, the derived unit *three feet*, and the number of derived units, *ten*. We have ten *threes*. To find the number of primary units we use multiplication, which gives thirty *ones*; the quantity is now more definite.

Again, in the quantity $5 \times \$3$, the primary unit is \$1, the derived (direct, actual) unit \$3, five of which=15 primary units.

The derived unit is not necessarily a *multiple* of the primary unit; it may be one or more of its *equal parts*. Thus in $\$\frac{5}{2}$, the primary unit is, as above, \$1, while the derived unit is $\$\frac{1}{2}$, the number of them *five*. The fraction $\frac{5}{2}$ expresses the ratio of the measured quantity ($\$\frac{5}{2}$) to the primary unit (\$1). The numerator shows how many derived units make up the quantity, the denominator shows the relation between the derived and primary units. It is thus seen that the fraction involves no new idea. Its notation is more complete than that of the integer in that it defines the derived unit—makes *explicit* what is implied in the integral notation. This appears in the processes of finding the value of 5 hats (*a*) at \$3 each, (*b*) at $\$\frac{1}{2}$ each.

$$5 \times \$3 = 5 \times \overline{3 \times \$1} = 15 \times \$1 = \$15.$$
$$5 \times \$\tfrac{1}{2} = 5 \times \overline{\tfrac{1}{2} \times \$1} = \tfrac{5}{2} \times \$1 = \$\tfrac{5}{2}.$$

The denominator 2 shows the relation between the derived unit ($\$\frac{1}{2}$) and the primary unit (\$1). In \$15, however, there is nothing to show the relation between \$3 and \$1. (This is seen in $5 \times \overline{3 \times \$1}$). In no other respect does the fraction differ from the integer. Both 15 and $\frac{5}{2}$ express ratio to the primary unit \$1. The 15 shows the number of primary units, but not that of the derived units. The $\frac{5}{2}$ shows both; there are 5 derived units, $\frac{5}{2}$ primary units.

In view of these facts it appears that a correct definition of *number* includes that of *fraction*, which is simply a number whose notation gives a more complete statement of the mental processes by which number is constituted. For mathematical purposes *Newton's* definition cannot be much improved: "Number is the abstract ratio of one quantity to another quantity of the same kind." Ratio being a pure abstraction, the word "abstract" should be omitted. Euler says, "Number is the ratio of one quantity to another quantity taken as unit." Drs. McLellan and Dewey define number as, "The repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind."*

*In conclusion I wish to say that every live teacher should read "*The Psychology of Number*,"

It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{3n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of rátio as is m. Hence each definition applies to fractions as well as integers.

It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the *unit* (1 egg); it is fractional with respect to the *unity* or whole (dozen)—5 out of 12, 5 *twelfths*. Five *half-yards* is just as integral as 5 *yards*. The *ratio* in each is *five*. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.

For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc $AX=x$ and arc $AY=y$. Take M at the middle of XY, and draw lines as indicated.

Then $DY=\sin y$, $HX=\sin x$, $EM=\sin\frac{1}{2}(x+y)$, $KY=\sin\frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos\frac{1}{2}(x+y)$, $CK=\cos\frac{1}{2}(x-y)$.

Now, $HX+DY=2KF=2EM\frac{KF}{EM}=2EM\frac{CK}{CM}=2EM.CK.$

That is, $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$............(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.

Again, $CH+CD=2CF=2CE.\frac{CF}{CE}=2CE\frac{CK}{CM}=2CE.CK$,

$$\text{or } \cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The triangles CEM and XNY are similar;

$$\text{hence } \frac{NX}{NY}=\frac{CE}{CM}, \text{ or } NX=2CE\frac{\tfrac{1}{2}XY}{CM}=2CE.KY,$$

$$\text{that is, } \sin x-\sin y=\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad\ldots\ldots\ \ldots\ldots(3).$$

Similarly, $\frac{NY}{XY}=\frac{EM}{CM}$, or $NY=2EM\frac{\tfrac{1}{2}XY}{CM}=2EM.KY$,

$$\text{or } \cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad\ldots\ldots\ldots\ldots\ldots(4).$$

Equation (1) can be made very useful in computing trigonometric tables, as the writer intends subsequently to show.

Now let $AM=x$ and $MY=MX=y$. Then $AY=x-y$ and $AX=x+y$. We have $(CM)^2-(CK)^2=(CY)^2-(CK)^2=(KY)^2$. But $\frac{CM}{ME}=\frac{CK}{KF}=\frac{KY}{LY}$.

Therefore $(ME)^2-(KF)^2=(LY)^2=(KY)^2-(KL)^2$,

or $(KF)^2-(KL)^2=(ME)^2-(KY)^2$,

or $(KF+KL)(KF-KL)=(ME)^2-(KY)^2$,

or $HX\times DY=(ME)^2-(KY)^2$.

$$\text{That is, } \sin(x+y)\sin(x-y)=\sin^2x-\sin^2y=\cos^2y-\cos^2x \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Again, $\frac{CM}{CE}=\frac{CK}{CF}=\frac{KY}{KL}$.

Therefore $(CE)^2-(CF)^2=(KL)^2=(KY)^2-(LY)^2$,

or $(CF)^2-(LY)^2=(CE)^2-(KY)^2$,

or $(CF-LY)(CF+LY)=CH\times CD=(CE)^2-(KY)^2$.

$$\text{That is, } \cos(x+y)\cos(x-y)=\cos^2x-\sin^2y=\cos^2y-\sin^2x \quad\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Let $DC=R$ the radius of a circle. Let the angle $CDB=2x$. Then $DAB=DBA=CBE=x$.

Then we have $\tan x=\frac{EC}{EB}$, also $\tan x=\frac{BE}{AE}$.

The product of these gives, $\tan^2x=\frac{CE}{AE}$, or $CE\times AE=(BE)^2$,

$$\text{or } \frac{EC}{AE}=\left(\frac{BE}{AE}\right)^2=\tan^2x.$$

Also, $\frac{EC}{BE}=\frac{\text{vers}2x}{\sin 2x}=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x}=\tan x$ [see above]........(7).

Then $1+\tan^2 x=1+\frac{EC}{AE}=\frac{AC}{AE}=\frac{2R}{AE}$,

$$1-\tan^2 x=1-\frac{EC}{AE}=\frac{AE-EC}{AE}=\frac{AC-2EC}{AE}=\frac{2(R-EC)}{AE}.$$

$\cot 2x=\frac{DE}{BE}$ and $\operatorname{cosec} 2x=\frac{R}{BE}$. From these values we at once have,

$\frac{2\tan x}{1+\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2R}=\frac{BE}{R}=\sin 2x$ ....(8).

$\frac{2\tan x}{1-\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2(R-EC)}=\frac{BE}{R-EC}=\frac{BE}{DE}=\tan 2x$ ....(9).

$\tan^2 x+2\cot 2x\tan x=\frac{EC}{AE}+\frac{2DE}{BE}\cdot\frac{BE}{AE}=\frac{EC+2DE}{AE}=\frac{AE}{AE}=1$ ....(10).

$2\operatorname{cosec} 2x\tan x-\tan^2 x=\frac{2R}{BE}\cdot\frac{BE}{AE}-\frac{EC}{AE}=\frac{2R-EC}{AE}=\frac{AE}{AE}=1$ ....(11).

$\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{2(R-EC)}{AE}\cdot\frac{AE}{2R}=\frac{R-CE}{R}=\frac{ED}{R}=\cos 2x$.....................(12).

$\operatorname{cosec} 2x-\cot 2x=\frac{R-ED}{BE}=\frac{EC}{BE}=\tan x=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x}$..............(13).

$\operatorname{cosec} 2x+\cot 2x=\frac{R+ED}{BE}=\frac{AE}{BE}=\cot x=\frac{\sin 2x}{1-\cos 2x}=\frac{1+\cos 2x}{\sin 2x}$..... ....(14).

$$\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x}=\frac{R+BE-ED}{R}\div\frac{R+BE+ED}{R}=\frac{EC+BE}{AE+BE}.$$

But $AE=\frac{(BE)^2}{EC}$; $\therefore\ \frac{CE+BE}{AE+BE}=\frac{EC+BE}{(BE)^2\div EC+BE}$

$$=\frac{EC(EC+BE)}{BE(EC+BE)}=\frac{EC}{BE}=\tan x\text{.................. ..(15).}$$

Again, $\cos x=\frac{AE}{AB}$, also $\cos x=\frac{AB}{AC}$.

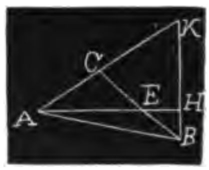
Twice the product of these gives $2\cos^2 x=\frac{2AE}{AC}=\frac{AE}{R}$.

Also $\cos 2x=\frac{DE}{R}$. $1+\cos 2x=\frac{DE+R}{R}=\frac{AE}{R}$. $\therefore 1+\cos 2x=2\cos^2 x$......(16).

$\sin x=\frac{CB}{AC}=\frac{BC}{2R}$; also $\sin x=\frac{EC}{BC}$. Twice the product of these gives

$2\sin^2 x=\frac{EC}{R}$. $1-\cos 2x=\frac{R-ED}{R}=\frac{EC}{R}$. $\therefore 1-\cos 2x=2\sin^2 x$..........(17).

To prove the "Tangent Proportion," let ABC be a plane triangle, the parts being represented as usual. Take $CE=CA$ and draw AEH. Draw BHK perpendicular to AH, to meet AC prolonged in K. Now considering the triangles ABC and ACE, the sum of the angles at A and E of the one is equal to the sum of the angles at A and B of the other. Hence $CAE+CEA=A+B$; and $CAE=CEA=BEH=\frac{1}{2}(A+B)$.



Also $BAE=A-\frac{1}{2}(A+B)=\frac{1}{2}(A-B)$. The angles at B and K of the triangle BCK are equal; for CBK is the complement of BEH or AEC, and BKC is the complement of the equal angle CAE. Hence $CK=CB=a$ and $AK=a+b$.

Now $\tan\frac{1}{2}(A-B)=\frac{BH}{AH}$ and $\tan\frac{1}{2}(A+B)=\frac{HK}{AH}$. $\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{BH}{HK}$.

But $\frac{BH}{HK}=\frac{BE}{AK}=\frac{a-b}{a+b}$. $\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}$...... ....(1).

From the triangle ABE, $\frac{BE}{AB}=\frac{\sin BAE}{\sin AEC}$, or $\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$..(2).

In the triangle AHK, $AH=AK\cos HAK=(a+b)\cos\frac{1}{2}(A+B)$.

In the triangle ABH, $AH=AB\cos BAH=c\cos\frac{1}{2}(A-B)$.

Equating, we have, $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$ ....(3).

Equation (3) divided by (2) also gives (1).

TWO PERPENDICULARS TO A TRANSVERSAL.

By JOHN N. LYLE, Ph. D., Bentonville, Arkansas.

Do two perpendiculars to a transversal intersect?

Both Euclid and Lobatschewsky affirm that they do not. Euclid regards the two perpendiculars as equidistant, whilst Lobatschewsky considers them as diverging.

Experience confirms the view that the distance between the perpendiculars is a constant. As long as this is the case it is evident that intersection is impossible. If the perpendiculars do not approach each other within the range of observation and experience what would anology and induction indicate? Would they not unmistakably favor the hypothesis that the perpendiculars do not intersect beyond the limits of observation and experience? Our knowledge of *the here* and *the now*, if at all accurate, must assuredly count for something *elsewhere* and *tomorrow*.

But aside from conclusions based on purely empirical data and obtained by analogical and inductive processes the assumption that a straight line that has a beginning and an end is *infinite* involves contradiction and is therefore absurd. One end of each perpendicular is at the transversal. If these perpendiculars intersect each of them has two ends. But *two* ends is the distinctive characteristic of a finite straight line.

The further assumption that the intersection takes place at a hypothetical place called "infinity" does not remove the difficulty in the slightest. *Two* ends are still attributed to the *supposed* infinite line.

There is in reality a new difficulty and a very serious one, for the logical law of non-contradiction is violated.

The difficulty is not that the human mind by reason of its limited powers is unable to cognize an unlimited straight line and discover what will or will not take place "at infinity," but it is that the mind by reason of the logical law of non-contradiction can not cognize a line that is at the same time both unlimited and limited.

As a result of this brief investigation we find that there are insuperable difficulties, logical, geometrical, and philosophical, in the hypothesis that two perpendiculars to a transversal intersect at a supposed place called "infinity."

Notwithstanding these difficulties in the way of this hypothesis many analysts daily and habitually accept it. They do make the "assumption that parallel lines, extended to an infinite distance, do intersect."

Euclid flatly contradicts this hypothesis in his statement that "parallels never meet however far they may be produced." In favor of Euclid's statement there is nothing in logic, science or geometry known to man that conflicts with it. I understand Mr. Drummond's protest to extend not only to Euclid's assumption but also to the assumption that Euclid contradicts.

If the analysts "can not comprehend the infinite" why do they employ the symbol of the infinite so freely in their equations and decide without hesitation so many questions against the Alexandrian geometer? The analysts make large use of the symbol ∞ in their equations. Do they or do they not comprehend the meaning of the symbolism employed? If they find ∞ incomprehensible, can they not obtain all legitimate results by the aid of *finite* quantities alone?

DEVELOPMENT OF SINθ AND COSθ.

By J. M. BANDY, Trinity College, Trinity, North Carolina.

In discussing the power of the calculus with my own students in Trinity College, I, several years ago, sprung the question "why can the trigonometric functions, sine and cosine, be developed by series?"

The calculus very readily furnished the series; but it did not expose the exponential nature of the functions.

The fact that the value of the functions can be expressed by series forced me to the conclusion that the reason existed in the nature of the functions themselves, and, therefore, they should yield this result directly.

Before proceeding to obtain the series directly from the functions, it will be necessary to produce a series involving an exponential function. The object thereafter will be to trace the law which connects sine and cosine with this exponential function.

We will develop $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ which gives us a simple converging series. This series can be made to express an exponential function.

Denoting $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ by e; that is, as x increases indefinitely, the *limiting value* of this function $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ is e.

$\therefore\ e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}$, etc.* From this we get

$$e^\theta=\left\{\left(1+\frac{1}{x}\right)^x\Big]_\infty\right\}^\theta=1+\theta+\frac{\theta^2}{1.2}+\frac{\theta^2}{1.2.3}+\text{etc.}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$e^{\frac{1}{x}}=\left\{\left(1+\frac{1}{x}\right)^x\Big]_\infty\right\}^{\frac{1}{x}}=1+\frac{1}{\infty}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

*This gives $e=2.71828$, the Naperian base.

and $\log\left(1+\frac{1}{\infty}\right)=\frac{1}{\infty}\log e$ ....(3).

To expose the principles which connect $\sin\theta$ and $\cos\theta$ with the above equations, and thus show that they can be expressed by series.

By geometry, $\cos^2\theta+\sin^2\theta=1$ ....(4).

The first member of (4) may be expressed thus: $\cos^2\theta-(-\sin^2\theta)=1$.

(4), therefore, becomes $\cos^2\theta-(-\sin^2\theta)=1$............................. (5).

Factoring first member of (5), we have,

$$(\cos\theta+\sin\theta\sqrt{-1})(\cos\theta-\sin\theta\sqrt{-1})=1 \ldots\ldots\ldots\ldots\ldots(6).$$

Taking log. of (6), we have $\log(\cos\theta+\sin\theta\sqrt{-1})+\log(\cos\theta-\sin\theta\sqrt{-1})=0$,

or $\log(\cos\theta+\sin\theta\sqrt{-1})=-\log(\cos\theta-\sin\theta\sqrt{-1})$.........................(7).

Denoting either member of (7) by y, we have,

$$\left.\begin{array}{r}\log(\cos\theta+\sin\theta\sqrt{-1})=y,\\ \text{and } \log(\cos\theta-\sin\theta\sqrt{-1})=-y,\end{array}\right\} \ldots\ldots\ldots\ldots\ldots(8).$$

$\therefore\ \cos\theta+\sin\theta\sqrt{-1}=10^y$, ...(9), and $\cos\theta-\sin\theta\sqrt{-1}=10^{-y}$.....(10).

Summing (9) and (10), $2\cos\theta=10^y+10^{-y}$.......................(11).

By trigonometry, $\cos^2\frac{1}{2}\theta=\frac{1}{2}(1+\cos\theta)=\frac{1}{4}(2+2\cos\theta)$

$$=\tfrac{1}{4}(10^y+2+10^{-y}), \text{ [from (11)]} \ldots\ldots\ldots\ldots(12),$$

and $-\sin^2\frac{1}{2}\theta=\frac{1}{2}(\cos\theta-1)=\frac{1}{4}(2\cos\theta-2)=\frac{1}{4}(10^y-2+10^{-y})$, [from (11)]....(13).

Extracting square roots of (12) and (13),

$$\cos\tfrac{1}{2}\theta=10^{\frac{y}{2}}+10^{-\frac{y}{2}}, \ldots\ldots\ldots\ldots\ldots(14),$$

$$\text{and } \sin\tfrac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}-10^{-\frac{y}{2}} \ldots\ldots\ldots\ldots\ldots(15).$$

Adding (14) and (15), $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}$....................(16).

Comparing (16) and (9), we see that θ may be changed into $\frac{1}{2}\theta$, provided that y is changed into $\frac{1}{2}y$. The same changes may, therefore, be made in (16): $\frac{1}{2}\theta$ may be changed into $\frac{1}{4}\theta$, if $\frac{1}{2}y$ is changed into $\frac{1}{4}y$. (16), therefore, becomes

$$\cos\tfrac{1}{4}\theta+\sin\tfrac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}} \ldots\ldots\ldots\ldots\ldots(17).$$

Repeating this change, we have, $\cos\frac{1}{4}\theta+\sin\frac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}}$ ....(18).

Thus we see that θ may be divided by any power of 2, however great, provided y is divided by the same power.

Let, then, $m=2^n$ ....(19).

We then have, $\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1}=10^{\frac{y}{m}}$ ....(20).

Taking log of (20), we have, $\log(\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1})=\frac{y}{m}$ ...... (21).

But when n in (19) becomes infinite, m becomes infinite.

$\therefore$ $\cos\frac{1}{m}\theta$ in the limit equals 1, and $\sin\frac{1}{m}\theta\sqrt{-1}$ in the limit equals the arc. $\therefore$ (21) becomes $\log(1+\frac{\theta}{m}\sqrt{-1})=\frac{y}{m}$ ....(22).

But from (3), (22) becomes $\frac{\theta}{m}\sqrt{-1}\log e=\frac{y}{m}$, or $y=\theta\sqrt{-1}\log e$ .....(23).

Substituting this value of y in (8), $\log(\cos\theta+\sin\theta\sqrt{-1})=\theta\sqrt{-1}\log e$..(24),

and $\log(\cos\theta-\sin\theta\sqrt{-1})=-\theta\sqrt{-1}\log e$..........(25).

Whence $\cos\theta+\sin\theta\sqrt{-1}=e^{\theta\sqrt{-1}}$..........(26),

and $\cos\theta-\sin\theta\sqrt{-1}=e^{-\theta\sqrt{-1}}$..........(27).

Adding (26) and (27), and dividing by 2, $\cos\theta=\frac{1}{2}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})$ ....(28),

by subtracting (27) from (26), and multiplying by $\sqrt{-1}$,

$$\sin\theta=-\tfrac{1}{2}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})\sqrt{-1}\ \ldots\ldots\ldots(29).$$

(28) and (29) enable us to develop $\cos\theta$ and $\sin\theta$ in a series arranged according to the powers of θ. Since $(\theta\sqrt{-1})^2=-\theta^2$, $(\theta\sqrt{-1})^3=-\theta^3\sqrt{-1}$, $(\theta\sqrt{-1})^4=\theta^4$, the substitution of $\theta\sqrt{-1}$ for θ in (1), gives

$$e^{\theta\sqrt{-1}}=1+\theta\sqrt{-1}-\frac{\theta^2}{1.2}-\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}+\frac{\theta^3\sqrt{-1}}{1.2.3.4.5}\ \ldots\ldots\ldots(30),$$

and
$$e^{-\theta\sqrt{-1}}=1-\theta\sqrt{-1}-\frac{\theta^2}{1.2}+\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}-\frac{\theta^3\sqrt{-1}}{1.2.3.4.5}\ \ldots\ldots\ldots(31).$$

Half the sum of (30) and (31) by (28) gives

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5.6} + \text{ etc.},$$

and half the difference of (30) and (31) by (29) gives

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^3}{1.2.3.4.5} - \text{ etc.}$$

The above are the required series. It is hoped that the law connecting $\cos\theta$ and $\sin\theta$ has been made plain.

(28) and (28) are Euler's results reached in a different way.

From (28) and (29) Demoivre's Theorem, which enables us to obtain the n roots of $y^n+1=0$ and $y^n-1=0$, is derived.

November 4, 1893.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6%?

III. Solution by the PROPOSER.

Let x = equated time.

Now the amount of \$100 for $(x-2)$ years + the present worth of \$200 due $(4-x)$ years hence must = \$300.

$100+6(x-2)$ = amount of \$100 for $(x-2)$ years at 6%.

$\frac{10000}{62-3x}$ = present worth of \$200 due $(4-x)$ years hence at 6%.

$$\therefore\ 100+6(x-2)+\frac{10000}{62+3x}=300.$$

$\therefore\ x=3.31533+$ years = 3 years, 3 months, 24 days.

PROOF. \$107.89 = amount of \$100 for 1.31533 years at 6%.

\$192.11 = present worth of \$200 due 0.68467 year hence at 6%.

\$107.89 + \$192.11 = \$300.

QUERY: Will the answers prove as obtained to the solutions of this problem on page 238, Vol. III.?

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The following method of solution I have found to be infallible for all problems of Compound Proportion.

The first ratio (simple) has for its antecedent the *quantity to be found*, and for its consequent the corresponding similar quantity of the problem; hence $x:70$.

We now reason as follows: Work, time, etc., as the case may be, being equal, can a *longer* or *shorter* ditch be dug—

(1). By digging it 40 feet wide than by digging it 25 feet wide? Evidently shorter; hence 25:40.

(2). By digging it 3 feet deep than by digging it 4 feet deep? Longer; hence 4:3.

(3). With 15 men than with 27 men? Shorter; hence 15:27.

(4). In 16 days than in 10 days? Longer; hence 16:10.

(5). By working 9 hours a day than by working 7 hours? Longer; hence 9:7.

(6). With \$500 than with \$375? Longer; hence 500:375.

$$\text{Whence, } x:70::\left\{\begin{array}{r}25:40\\ 4:3\\ 15:27\\ 16:10\\ 9:7\\ 500:375\end{array}\right. \qquad \therefore x=88\tfrac{8}{9}.$$

Solved with same result by *P. S. BERG* and *EDWARD R. ROBBINS*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

There are two interpretations of the problem.

(1). The men are paid by the cubic foot; in this case the second lot should handle $\frac{5}{3}\frac{0}{7}\frac{0}{5}=\frac{4}{3}$ as much dirt as the first lot.

$$\therefore \tfrac{4}{3}\times\frac{70\times25\times4}{40\times3}=\tfrac{700}{9}=77\tfrac{7}{9} \text{ rods length of ditch.}$$

(2). Both ditches are dug by contract and the men are worked at their best all the time; in this case the amount received has nothing to do with the length of the ditch.

$$\therefore\left\{\begin{array}{c}27\\10\\7\end{array}\right\}:\left\{\begin{array}{c}70\\25\\4\end{array}\right\}=\left\{\begin{array}{c}15\\16\\9\end{array}\right\}:\left\{\begin{array}{c}x\\40\\3\end{array}\right\}.$$

$$\therefore x=\frac{70\times25\times4\times15\times16\times9}{27\times10\times7\times40\times3}=66\tfrac{2}{3} \text{ rods.}$$

[NOTE. $88\frac{8}{9}$ is obtained by multiplying $66\frac{2}{3}$ by $\frac{4}{3}$.]

Solved with same result as in (1) by *FREDERICK R. HONEY.*

[NOTE. There seems to be some disagreement among our contributors as to the correct solution of this problem. I, however, agree with Mr. Gruber, and have used his method of solution for several years. For a more detailed statement of this method the reader is referred to my *Mathematical Solution Book*. EDITOR.]

65. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at \$5.37½ per yard, less 12½ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at 5½%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

I. Solution by P. S. BERG, Larimore, North Dakota, and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

$1.00-.12\frac{1}{2}=.87\frac{1}{2}$, $1.00-.10=.90$. $\therefore 1.00\times.87\frac{1}{2}\times.90=78\frac{3}{4}\%$.

$\$5.37\frac{1}{2}\times.78\frac{3}{4}\times250=\$1058.203125=$ cost.

From April 4th to September 5th is 5 months, 1 day, at 6%, \$1. amounts to $\$1.025\frac{1}{6}$. $\$1058.203125\times1.025\frac{1}{6}=\$1084.8336$.

$1.00+.15=1.15$, $1.00+.10=1.10$, $1.00+.05=105\%$.

$1.15\times1.10\times1.05=132.825\%$.

$\$5.37\frac{1}{2}\times1.32825\times250=\$1784.8359375$.

From October 19th to December 8th, 50 days, at $5\frac{1}{2}\%$, $\$1.=.007\frac{2}{3}\frac{5}{6}$.

$\$1.00-\$.007\frac{2}{3}\frac{5}{6}=\$.992\frac{1}{3}\frac{1}{6}$.

$\$1784.8359375\times.992\frac{1}{3}\frac{1}{6}=\$1771.2017$.

$\$1771.2017-\$1084.8336=\$686.3681$ profit.

PROBLEMS.

69. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats. 13 Farmers; 3 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrats and Farmers. Required the number of men in the group.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years; when can I pay him \$100 to settle the account equitably, money being worth 6%?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the loci of the foci of variable ellipses passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having for diameters the semi-latas rectum of the fixed ellipse.

Solution by the PROPOSER.

If the given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(1),

the equation to the required ellipse is of the form $\frac{x^2}{a_1{}^2}+\frac{(y-n)^2}{b_1{}^2}=1$ ....(2).

This passing through $(ae, 0)$, we have $\frac{a^2e^2}{a_1{}^2}+\frac{n^2}{b_1{}^2}=1$ ....(3).

The directrix of (2) is $x=\frac{a_1}{e_1}$..(4), e_1 being the eccentricity of (2), and $x=a$..(5) is the tangent to the given ellipse at the extremity of its major axis. Then

$\frac{a_1}{e_1}=a$............(6), or $a_1=ae_1$............(7), $a_1e_1=ae_1{}^2$...........(8).

Let (x', y') be the coordinates of the right hand focus of (2) in any one of its positions; then $a_1e_1=x'$.... ......(9), $n=y'$............(10), and by (8) and (9), $e_1{}^2=\frac{x'}{a}$, $1-e_1{}^2=\frac{a-x'}{a}$ ....(11).

Also by (7), $a_1{}^2=a^2e_1{}^2=ax'$. ....(12);

$\therefore$ $b_1{}^2=a_1{}^2(1-e_1{}^2)=x'(a-x')$.......................(13), and (3) becomes

$\frac{a^2e^2}{ax'}+\frac{y'^2}{x'(a-x')}=1$ ....(14).

Reducing, $x'^2+y'^2-a(1+e^2)x'=-a^2e^2$............(15), a circle whose center is on the axis of x, passing through $(a, 0)$, and having diameter $\frac{b^2}{a}$.

Also solved by *G. B. M. ZERR*.

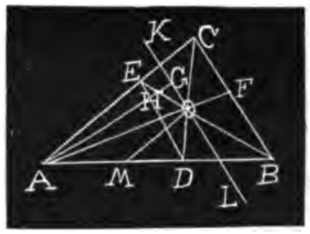
61. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O and meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the escribed circles.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates, equation to CD is $\alpha-\beta=0$; to BE, $\alpha-\gamma=0$.

$$\therefore \left(\frac{2\triangle}{a+b}, \frac{2\triangle}{a+b}, 0\right), \left(\frac{2\triangle}{a+c}, 0, \frac{2\triangle}{a+c}\right),$$



are the coordinates of D, E.

$\therefore$ $\beta+\gamma-\alpha=0$, is the equation to DE.

The distance from O, (r, r, r) from this line is,

$$p_1=\frac{r}{\sqrt{3abc+2\cos C-2\cos A+2\cos B}}$$

$$=\frac{r}{\sqrt{\dfrac{3abc+a^2c+b^2c-c^3-ab^2-ac^2-b^3+a^2b+bc^2+a^3}{abc}}}$$

$$=\frac{r}{\sqrt{\dfrac{abc+(a+b+c)(a+b-c)(a-b+c)}{abc}}}=\frac{r}{\sqrt{\dfrac{abc+8s(s-b)(s-c)}{abc}}}$$

$$=\frac{r}{\sqrt{\dfrac{abc(s-a)+8\triangle^2}{abc(s-a)}}}=\frac{r}{\sqrt{\dfrac{\dfrac{abc}{4\triangle}+\dfrac{2\triangle}{s-a}}{\dfrac{abc}{4\triangle}}}}=\frac{r}{\sqrt{\dfrac{R+2r_1}{R}}}=\frac{rR}{\sqrt{R^2+2Rr_1}}=\frac{rR}{d_1}.$$

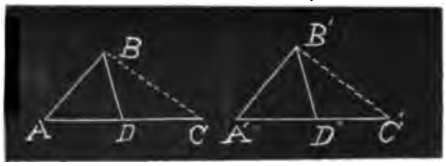
Similarly $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median of one of them* equal, each to each.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland, and CHAS. C. CROSS, Laytonsville, Maryland.

Let $AB=A'B'$, $AC=A'C'$, $BD=B'D'$. $\triangle ABD=\triangle A'B'D'$, because all the sides are equal, each to each.



Then $\triangle BDC=\triangle B'D'C'$, having two sides and included angle of one= two sides and included angle of the other.

$\therefore$ $\triangle ABC=\triangle A'B'C'$.

Also solved by *EDWARD R. ROBBINS*, *M. A. GRUBER*, and *G. B. M. ZERR*.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone *if the angle at its vertex is less than a right angle.*

Solution by the PROPOSER.

Let the base and the axis of the cone coincide with the xy-plane and the z-axis respectively. Then if c denote the altitude of the cone and ϕ the angle which any one of its elements makes with the base, its equation is

$$(x^2+y^2)\tan^2\phi=(z-c)^2.$$

The equation of a plane through the y-axis and inclined at an angle θ to the xy-plane is

$$z=x\tan\theta.$$

The projection on the xy-plane of the intersection of the two surfaces is

$$(x^2+y^2)\tan^2\phi=(x\tan\theta-c)^2=x^2\tan^2\theta-2cx\tan\theta+c^2.$$

This becomes, when referred to rectangular axes in the plane of the section, the origin and y-axis being unchanged, $(x^2\cos^2\theta+y^2)\tan^2\phi=x^2\sin^2\theta-2cx\sin\theta+c^2$, or $x^2(\cos^2\theta\tan^2\phi-\sin^2\theta)+y^2\tan^2\phi+2cx\sin\theta-c^2=0$, which represents a rectangular hyperbola if $\tan^2\phi+\cos^2\theta\tan^2\phi-\sin^2\theta=0$. From this equation,

$$\sin^2\theta=\frac{2\tan^2\phi}{\tan^2\phi+1}=2\sin^2\phi, \text{ and } \sin\theta=\pm\sqrt{2}\sin\phi.$$

Since $\sin\phi$ cannot be greater than $\frac{1}{\sqrt{2}}$, ϕ cannot exceed 45°. Hence the angle at the vertex of the cone cannot be less than 90°.

Other solutions of this problem will appear in next issue.

PROBLEMS.

67. Proposed by F. M. PRIEST, St. Louis, Mo.

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago; Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with A_1, A_2, A_3, ...$A_{\frac{n-1}{2}}$, where n is supposed to be odd.

Prove that $OA_1 - OA_2 + OA_3 - OA_4 + \ldots \pm OA_{\frac{n-1}{2}}$, every other chord being affected with the minus sign.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A vertical slit is made in the middle of the side of a rectangular box containing water. What is the time required to empty the box?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let a, b, h=length, width, and depth of box, c=width of slit, m=coefficient of contraction, z=distance of surface of water from bottom of box, x=distance of any elemental area of slit from bottom of box.

$\therefore$ The quantity discharged through the slit in an element of time is

$$Q=[mc\sqrt{2g}\int_0^z \sqrt{z-x}\,dx]dt=\tfrac{2}{3}mc\sqrt{2g}\,z^{\frac{3}{2}}\,dt=abdz.$$

$$\therefore\ t=\frac{3ab}{2mc\sqrt{2g}}\int_{h'}^{h}\frac{dz}{z^{\frac{3}{2}}}=\frac{3ab(\sqrt{h}-\sqrt{h'})}{mc\sqrt{2ghh'}},\ \text{for depth } (h-h').$$

When $h'=0$, t=infinity or it is impossible to absolutely empty the box.

II. Solution by the PROPOSER.

Let x=distance from base of box to any point in the vertical slit below surface of water.

Let y=distance from base of box to surface of water.

The velocity of discharge for point $x=\sqrt{2g(y-x)}$.

$\therefore\ dF=k\sqrt{2g(y-x)}dx$, where k=width of slit, and F=flow of water.

Whence $F=k\sqrt{2g}\int_0^y \sqrt{y-x}\,dx=\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

Call V the volume of water in the box at any instant.

Then $\frac{dV}{dt}=-\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$. But $V=aby$, where a and b are the dimensions of base of box. $\therefore\ \frac{abdy}{dt}=-\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

From which $t=-\frac{3ab}{2k}\int_n^m y^{-\frac{3}{2}}dy=\frac{ab}{k\sqrt{2g}}\left[\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{m}}\right]$, m and n being the depths of water at beginning and end of time of discharge.

If $n=0$, or the box is emptied, $t=\infty$.

If $m=\infty$, $t=\frac{ab}{k\sqrt{2gn}}$; or the time to empty a box of infinite depth to a finite depth is finite.

37. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

A thin board, of which the elements are given, is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Take the horizontal line through the point making the greatest angle with the plane in its initial position as the axis of x, and the axis of y vertically downward through the same point. Let R and T be the normal and tangential reactions of the plane and sphere at any time t from the commencement of motion, θ and ϕ the angles of rotation of the sphere and of the plane, m, k, a the mass, radius of gyration, and radius of the sphere, and r=the distance the sphere has moved on the plane, and x and y the coordinates of the center of the sphere.

Resolving horizontally and vertically, and taking moments about the center of the sphere,

$m\frac{d^2x}{dt^2}=R\sin\phi-T\cos\phi$.........(1), $m\frac{d^2y}{dt^2}=mg-R\cos\phi-T\sin\phi$.........(2).

$mk^2\frac{d^2\theta}{dt^2}=Ta$..............(3). We also have $\theta=\frac{r}{a}+\phi$.............(4),

$x=r\cos\phi+a\sin\phi$................(5), and $y=r\sin\phi-a\cos\phi$...............(6).

Eliminating T from (1) and (2), $\sin\phi\dfrac{d^2x}{dt^2}-\cos\phi\dfrac{d^2y}{dt^2}=\dfrac{R}{m}-g\cos\phi$ .... (7).

Eliminating T and R from (1), (2), and (3),

$$\cos\phi\frac{d^2x}{dt^2}+\sin\phi\frac{d^2y}{dt^2}=g\sin\phi-\frac{k^2}{a}\frac{d^2\theta}{dt^2} \quad\ldots\ldots\ldots\ldots(8).$$

From (4), $\dfrac{d^2\theta}{dt^2}=\dfrac{1}{a}\dfrac{d^2r}{dt^2}+\dfrac{d^2\phi}{dt^2}$ ....(9).

From (5) and (6), $\dfrac{d^2x}{dt^2}=\cos\phi\dfrac{d^2r}{dt^2}-2\sin\phi\dfrac{dr}{dt}\dfrac{d\phi}{dt}-r\cos\phi\dfrac{d\phi^2}{dt^2}$

$$-r\sin\phi\frac{d^2\phi}{dt^2}-a\sin\phi\frac{d\phi^2}{dt^2}+a\cos\phi\frac{d^2\phi}{dt^2}\quad\ldots\ldots\ldots(10),$$

$$\frac{d^2y}{dt^2}=\sin\phi\frac{d^2r}{dt^2}+2\cos\phi\frac{dr}{dt}\frac{d\phi}{dt}-r\sin\phi\frac{d\phi^2}{dt^2}$$

$$+r\cos\phi\frac{d^2\phi}{dt^2}+a\cos\phi\frac{d\phi^2}{dt^2}+a\sin\phi\frac{d^2\phi}{dt^2}\quad\ldots\ldots\ldots(11).$$

Eliminating $\dfrac{d^2x}{dt^2}$, $\dfrac{d^2y}{dt^2}$, $\dfrac{d^2\phi}{dt^2}$ from (7) and (8),

$$2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}+a\frac{d\phi^2}{dt^2}=g\cos\phi-\frac{R}{m}\quad\ldots\ldots(12),$$

$$\frac{a^2+k^2}{a^2}\frac{d^2r}{dt^2}-r\frac{d\phi^2}{dt^2}+\frac{a^2+k^2}{a^2}\frac{d^2\phi}{dt^2}=g\sin\phi\quad\ldots\ldots(13).$$

(12) and (13) seem to indicate that one more condition at least should be given.

PROBLEMS.

46. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket
Ninety times as high as the moon." *Mother Goose.*

Neglecting the resistance of the air, how long did it take the old lady to go up?

47. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

What is the focus of the convex surface of a plano-convex lens, index μ, which will converge parallel monochromatic rays to a given focus, the rays entering the lens on the plane side?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Solve the equation $x^3+y^2=a^2$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $y=\dfrac{x(x-n^2)}{2n}$. Then we readily obtain $x^3+\left\{\dfrac{x(x-n^2)}{2n}\right\}^2=\left\{\dfrac{x(x+n^2)}{2n}\right\}^2$, which is a general formula for finding *the sum of a cube aud a square equal to a square*, x and n representing any values. We have also the general condition, derived from the formula, $nx+y=a$. By taking $n=1$, and putting $x=$, consecutively, the natural numbers beginning with unity, we obtain a series of equations in which the consecutive values both of y and a form *the series of integral numbers the sum of any two consecutive terms of which is the square of their difference.* [Problem 43, page 370, Vol. II.]

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $y=mx$, then $x^3+m^2x^2=a^2$. $\therefore$ $x+m^2=a^2/x^2=b^2$, $\therefore$ $b^2-m^2=x$, where b and m can be any integers $b>m$. We append some values.

b	m	x	y	a
1	0	1	0	1
2	1	3	3	6
3	2	5	10	15
4	3	7	21	28
5	4	9	36	45
&c.	&c.	&c.	&c.	&c.

III. Solution by M. C. STEVENS, M. A., Department of Mathematics, Purdue University, Layfayette, Indiana.

If x be any integer and $y=\dfrac{x(x-1)}{2}$, then $x^3+y^2=\dfrac{x^4+2x^3+x^2}{4}=a^2$.

$\therefore$ $a=\dfrac{x(x+1)}{2}$. If $x=1$, then $a=1$. If $x=2$, then $a=3$, and so on.
$y=0$ $y=1$

IV. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

We write $x^3=a^2-y^2$. From the well known form

$mn=\left(\dfrac{m+n}{2}\right)^2-\left(\dfrac{m-n}{2}\right)^2$, if $x^3=mn$, the problem is answered.

Let m and n be 4 and 2; or 27 and 1; or 9 and 3; etc.; then $2^3+1^2=3^2$; $3^3+13^2=14^2$; $3^3+3^2=6^2$; etc.

V. Solution by H. C. WILKES, Skull Run, West Virginia.

$x^3=(a+y)(a-y)$. Let $a+y=x^2$ and $a-y=x$, then $x^2+x=2a$, and $x=\frac{1}{2}\pm\sqrt{2a+\frac{1}{4}}$. Let a be any triangular number, and from the above formula, integral values for x, a, and y can be found.

VI. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let $x=ky$. Then $x^3+y^2=a^2$ becomes $y^2\{k^3y+1\}=a^2$. This will be a square if $y=k^3+2$. $\therefore$ $y=k^3+2$, and $x=k(k^3+2)$ will be a solution, where k is any integer. If $k=1$, $y=3$, $x=3$ and $x^3+y^2=36$. If $k=2$, $y=10$, $x=20$, and $x^3+y^2=8100$, etc., etc.

VII. Solution by J. H. DRUMMOND, LL. D., Portland, Maine.

(A). If the problem is to be taken literally, $y=\sqrt{a^2-x^3}$ in which x may any number whose third power $<$ than a^2. But this does not give exact results.

(B). If it means that $x^3+y^2=\square$, let $x=my$ and we have $m^3y+1=\square=$ (say) b^2 and $y=(b^2-1)/m^3$ and $x=(b^2-1)/m^2$; but then $a=b(b^2-1)/m^3$, in which m and b may be any numbers greater than unity, but the value of a depends on x and y.

(C). By transposing $x^3=a^2-y^2$; take $x=a-y$, then $x^2=a+y$, and $a^2-2ay+y^2=a+y$, and $y=(2a+1\pm\sqrt{8a+1})/2$. As y must be less than a to make x positive, the sign of the radical term must be negative. It is readily seen that $a=n(n+1)/2$ makes $8a+1$ a square, and by reducing we get $y=n(n-1)/2$ and $x=n$, in which n may be any number.

(D). If the question means to find exact values of x and y for *any* value of a, I cannot solve it.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In $x^2+x\sqrt{xy}=a$......(1) and $y^2+y\sqrt{xy}=b$......(2) find such values of a find b as will make x and y integral; give a general solution.

I. Solution by the PROPOSER.

Take $y=m^2x$, and by combining the two equations and reducing we have, $\frac{b}{a}(m+1)=m^3(m+1)$ and consequently $m^3=\frac{b}{a}$.

From (1) we have $x=\pm\sqrt{\frac{a}{m+1}}$. Take $a=c^2$ and $m+1=d^2$ and substituting, we have $x=c/d$. To make this value integral, take $c=de$; then $x=e$, and $y=m^2x=e(d^2-1)^2$. But $a=c^2$, and $c=dx=de$. $\therefore$ $a=d^2e^2$; but $b=am^3=d^2e^2(d^2-1)^3$, in which a may be any whole number>1, and e any whole number.

II. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

In order that $\sqrt{xy}$ be integral and rational, we put $x=rm^2$ and $y=rn^2$, r, m, and n being any integers. Whence we readily find that when $a=r^2m^3(m+n)$ and $b=r^2n^3(m+n)$, x and y are integral.

Now put $r=1$, $m=3$, and $n=2$, and we obtain $x^2+x\sqrt{xy}=135$ and $y^2+y\sqrt{xy}=40$; whence $x=9$ and $y=4$.

Put $r=2$, $m=2$, and $n=1$; then $x^2+x\sqrt{xy}=96$ and $y^2+y\sqrt{xy}=12$; whence $x=8$, and $y=2$.

III. **Solution by A. H. BELL, Box 184, Hillsboro, Illinois.**

The only condition to fill is to make $xy=\square$. Take $x=4$, $y=1$, and $a=24$, $b=3$, etc., etc.

IV. **Solution by H. C. WILKES, Skull Run, West Virginia.**

Let $m^2=x$, $n^2=y$. Then $m^3(m+n)=a$; $n^3(m+n)=b$. $\therefore$ To make x and y integral, a and b must have a common factor $(m+n)$. The remaining factors will be m^3 and n^3. Let $a=448$, $b=189$; then $x=16$, $y=9$; $7(64)m=4$; $7(27)n=3$.

V. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let $P=x^{\frac{1}{2}}$, $Q=y^{\frac{1}{2}}$. Then $P^4+P^3Q=a$......(1). $Q^4+Q^3P=b$.....(2).

$$(1)\div(2).\ P\div Q=\sqrt[3]{a}\div\sqrt[3]{b}.\quad \therefore P^2=x=\pm\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}},\ Q=y=\pm\frac{b^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}}.$$

Let $a=\{\frac{1}{2}(m^2+n^2)\}^3$, $b=\{\frac{1}{2}(m^2-n^2)\}^3$.

$$\therefore x=\pm\frac{(m^2+n^2)^2}{4m},\quad y=\pm\frac{(m^2-n^2)^2}{4m}.$$

$$\text{Let } m=pn.\quad \therefore x=\pm\frac{n^3(p^2+1)^2}{4p},\quad y=\pm\frac{n^3(p^2-1)^2}{4p}.$$

Let $n=2p$. $\therefore x=\pm2p^2(p^2+1)^2$, $y=\pm2p^2(p^2-1)^2$.
$\therefore a=\{2p^2(p^2+1)\}^3$, $b=\{2p^2(p^2-1)\}^3$.

VI. **Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

Let $y=m^2x$. Then $x^2(1+m)=a$, and $x^2m^3(1+m)=b$.

$$\therefore m=\frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}},\quad x=\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}};\quad y=\frac{b^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}}.$$

$$\text{Let } a=p^3;\ b=q^3.\quad \text{Then } x=\frac{p^2}{\sqrt{p+q}};\ y=\frac{q^2}{\sqrt{p+q}}.$$

Let $p=2rs$; $q=r^2+s^2$. Then $x=\dfrac{4r^2s^2}{r+s}$; $y=\dfrac{(r^2+s^2)^2}{r+s}$.

Let $r=k+l$; $s=k-l$. Then $x=\dfrac{2(k^2-l^2)^2}{k}$; $y=\dfrac{2(k^2+l^2)^2}{k}$.

Let $l=ak$. Then $x=2k^3(1-a^2)^2$; $y=2k^3(1+a^2)^2$.

Now $a=p^3=8r^3s^3=8(k^2-l^2)^3=8k^6(1-a^2)^3$, and $b=q^3=(r^2+s^2)^3=8(k^2+l^2)^3=8k^6(1+a^2)^3$, where a and k are integers.

PROBLEMS.

53. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Given $x^2-114\frac{1}{4}y^2=\mp3$ to find the least values of x and y in integers.

54. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the distance of two points within a square shall not exceed a side of the square. [From *Byerly's Integral Calculus.*]

I. Solution by ALWYN C. SMITH, The University of Colorado, Boulder, Colorado.

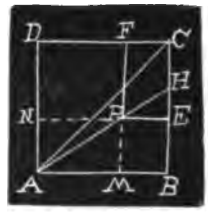
a is one side of the square; P and Q the two points; (x, y) the point P with O for origin; and r and ϕ the polar coordinates of Q, with P as origin. Then the favorable cases are

$$4\int_0^{\frac{1}{2}\pi}\int_0^a\int_0^{a-r\sin\phi}\int_0^{a-r\cos\phi}dx\,dy\,r\,dr\,d\phi=a^4(\pi-\tfrac{13}{6}).$$

All the cases $=a^2.a^2=a^4$. Therefore, $p=\pi-\frac{13}{6}$.

II. **Solution by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.**

Let a=a side of the square $ABCD$, and join A with any point P within the given square. Then as AP represents the distance and direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.



Let $AP=x$, $AH=x'$, and $\angle PAB=\theta$.

When $x'=a\sec\theta$, $PF=a-x\sin\theta$, $PE=a-x\cos\theta$.

$\therefore$ Area $PECF=(a-x\sin\theta)(a-x\cos\theta)$.

Hence the required chance is

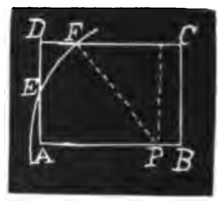
$$p_1=\frac{\int_0^{\frac{1}{4}\pi}\int_0^{a}(a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}{\int_0^{\frac{1}{4}\pi}\int_0^{x'}(a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}$$

$$=\frac{8}{a^4}\int_0^{\frac{1}{4}\pi}\int_0^{a}(a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx$$

$$=\tfrac{2}{3}\int_0^{\frac{1}{4}\pi}(6-4\sin\theta-4\cos\theta+3\sin\theta\cos\theta)d\theta=\pi-\tfrac{13}{6}.$$

III. **Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.**

Take a rectangle $ABCD$, with sides $AB=b$, and $BC=a$, such that a is not greater than b; and consider the chance that the proposed distance shall exceed b. Let N be the number of favorable cases; then if a be increased infinitesimally, dN will be the number of new cases introduced by placing each point in turn on the differential slice along b while the other one traverses the mixtilinear area DEF.



That is, taking AP equal to x,

$$dN=4\left[\int_{\sqrt{b^2-a^2}}^{b}\left(ax-\frac{a}{2}\sqrt{b^2-a^2}-\frac{x}{2}\sqrt{b^2-x^2}\right.\right.$$

$$\left.\left.-\frac{b^2}{2}\sin^{-1}\frac{x}{b}+\frac{b^2}{2}\cos^{-1}\frac{a}{b}\right)dx\right]da$$

$$=2[2ab-ab\sqrt{b^2-a^2}-\frac{a^3}{3}-\frac{\pi b^3}{2}+b^3\cos^{-1}\frac{a}{b}]da\text{; and,}$$

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^4}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}]+C.$$

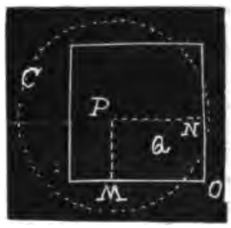
Since $N=0$ when $a=0$, $C=\frac{4b^4}{3}$; and,

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^2}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}+\frac{2b^4}{3}].$$

If now $a=b$, $N=b^4(\frac{19}{6}-\pi)$; and the whole number of cases $=b^4$. Hence the chance that the proposed distance shall exceed b is $\frac{19}{6}-\pi$; therefore, the chance that it will not exceed b is $\pi-\frac{13}{6}$.

IV. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Let a be one side of the square, and O the origin. With center O and radius a describe a quadrant. Let P any point within the square (x, y) be one point, and Q be the other point. With center P and radius a describe the circle C. Now Q may be anywhere within the area common to this circle and the square. The favorable cases may then be found by confining Q within the rectangle xy while P traverses the entire square, and then taking four times the result. Hence,



$$p=\frac{4}{a^4}\left\{\int_0^a\int_0^{\sqrt{a^2-y^2}}xydxdy+\tfrac{1}{2}\int_0^a\int_{\sqrt{a^2-y^2}}^a[x\sqrt{a^2-x^2}\right.$$

$$\left.+y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{x}{a}-a^2\cos^{-1}\frac{y}{a}]dxdy\right\}=\pi-\frac{13}{6}.$$

V. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

This problem affords a splendid test of the correctness of the general value for any convex area as demonstrated in problem 25, page 281, September-October MONTHLY.

Let AK, $AL=p$, $\angle LAB=\theta$, EF, $GH=C$.

For EF, $C=a\sec\theta$; the limits of p are $a\sin\theta$ to $a\cos\theta$.

For GH, $C=p\sec\theta\operatorname{cosec}\theta$; the limits of p are $a\sin\theta\cos\theta$ to $a\sin\theta$. The limits of θ are 0 to $\frac{1}{4}\pi$.

From problem 25,

$$\triangle=\frac{1}{3A^2}\int\int(C^3-3a^2C+2a^3)d\theta dp.$$

$$\therefore\ \triangle=\frac{8}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta\cos\theta}^{a\sin\theta}(p^3\sec^3\operatorname{cosec}^3\theta-3a^2p\sec\theta\operatorname{cosec}\theta+2a^3)d\theta dp$$

$$+\frac{4}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta}^{a\cos\theta}(a^3\sec^3\theta-3a^3\sec\theta+2a^3)d\theta dp.$$

$$\triangle=\tfrac{2}{3}\int_0^{\frac{1}{4}\pi}(\tan\theta\sec^2\theta-3\sin\theta\cos\theta-6\tan\theta+8\sin\theta)d\theta$$

$$+\tfrac{4}{3}\int_0^{\frac{1}{4}\pi}(\sec^2\theta-\tan\theta\sec^2\theta-3+3\tan\theta+2\cos\theta-2\sin\theta)d\theta.$$

$\therefore\ \triangle=\frac{19}{6}-\pi.\quad p=1-\triangle=\pi-\frac{13}{6}=$required chance.

PROBLEMS.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

45. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

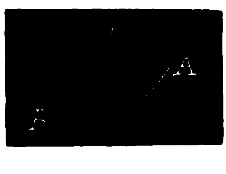
SOLUTIONS OF PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42^\circ 30'$ north$=\lambda$, at what angle with the horizon will the sun rise, its declination$=22^\circ$ north$=\delta$?

I. Solution by the PROPOSER.

Let BA be a portion of the equator, $CA=\delta$, a portion of a meridian passing through the sun at C when rising, and describing a small-circle arc CE, parallel with BA, and let BC be a portion of the horizon. Then the angles ECA, and BAC, each$=90^\circ$, because meridians cut the equator and circles of declination at right angles. Now $CBA=90^\circ-\lambda$, then $\sin BCA=\sin\lambda\sec\delta=\cos BCE$. $\therefore\ BCE=43^\circ 13' 37''=$required angle.



II. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let $x^2+z^2=R^2$..............(1) be the equation to the horison. Then, $x\cos\lambda+y\sin\lambda=R\sin\delta$..............(2) is the equation to the plane of the sun's path. (1) and (2) intersect in the points

$$x=\frac{R\sin\delta}{\cos\lambda},\quad y=0,\quad z=\pm\frac{R}{\cos\lambda}\sqrt{\cos^2\lambda-\sin^2\delta}.$$

The equation for the tangent plane for z positive is,

$$x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda.$$

$\therefore$ We must find the angle ϕ between the two lines in space,

$$\left.\begin{array}{r}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ x\cos\lambda+y\sin\lambda=R\sin\delta\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\left.\begin{array}{r}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ y=0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let $s=\dfrac{\sin\delta}{\sqrt{\cos^2\lambda-\sin^2\delta}}$, $t=\cot\lambda$.

Then $\cos\phi=\dfrac{1+s^2}{\sqrt{(1+s^2)(1+s^2+t^2)}}=\sqrt{\dfrac{1+s^2}{1+s^2+t^2}}$. $\quad\therefore\ \cos\phi=\dfrac{\sin\lambda}{\cos\delta}$.

$\therefore\ \phi=43°13'37''$.

III. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

Let HO represent the horizon, Z the zenith of the place of observation, EQ the equator, P the north pole, DL the diurnal circle of the sun, and S the position of the sun at rising. In the quadrantal spherical triangle ZPS, we have $ZS=90°$, $ZP=90°-\lambda$, $PS=90°-\delta$.

We find $\cos ZSP=\sin\lambda/\cos\delta$, but $\angle ZSP$ is equal to the angle which a tangent at S of the circle DSL makes with the horizon.

$\therefore\ \cos x=\dfrac{\sin\lambda}{\cos\delta}$, denoting by x the required angle. For the given concrete values we find $x=43°14'$.

[NOTE.—Prof. H. C. Whittaker obtained as the numerical result for the required angle 70 degrees, 21 minutes, 56 seconds, and Prof. E. W. Morrell obtained 72 degrees, 9 minutes, 4.2 seconds. The result given in the published solutions seems to us to be the correct one.]

39. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minute., makes 6000 vibrations in 1 hour and 9½ minutes. What is the length of the pendulum? And what length should it have to keep true time?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Regarding 1 hour, 13 minutes and 1 hour, 9½ minutes as registered by a clock keeping correct time, $g=32.16$, $\pi=3.1416$, $t=\pi\surd(l/g)$. Then 1 hour, 9½ minutes=4170 seconds.

$$\therefore t=\tfrac{4170}{6000}=\tfrac{139}{200}=\pi\sqrt{\frac{l}{g}}.\quad \therefore l=\frac{(139)^2g}{(200\pi)^2}=1.57393 \text{ ft.}=18.88716 \text{ inches.}$$

1 hour, 13 minutes=4380 seconds.

$$\frac{4380\times200}{139}=\text{number of vibrations in 1 hour, 13 seconds.}$$

$$\therefore \frac{4380\times200}{139}=4386 \text{ seconds.}$$

$$\therefore t'=\frac{4386\times139}{4880\times200}=\frac{731\times139}{730\times200}=\pi\sqrt{\frac{l'}{g}}.$$

$$\therefore l'=\frac{(731\times139)^2g}{(730\times200\pi)^2}=1.578243 \text{ feet.}$$

$$\therefore l'=18.93892 \text{ inches}=\text{length to keep true time.}$$

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

1 hour and 9½ minutes=4170 seconds. 4170 seconds÷6000=.695 seconds, the time of one vibration. From Mechanics $l=t^2g/\pi^2$, whence $l=18.886$ inches, the length of the pendulum. Again, 1 hour and 13 minutes = 4380 seconds. 4380÷.695=876/.139=number of vibrations in 1 hour and 13 minutes. As the pendulum gains 6 seconds in that time, 6÷(876/.139)=.834/876=.0095, the time in seconds gained in one vibration.

∴ .695 seconds+.0095 seconds=.69595 seconds, the time of vibrations of pendulum to keep correct time. Hence by substitutions in the above formula $l=18.9379$ inches, the length of pendulum to keep true time.

[Note.—The results sent in with the problem by the Proposer were, 18.89835+ inches, and for true time .036036+ inches longer. Prof. P. S. Berg in his solution obtained for length of pendulum 18.837975 inches, and 22.393 inches as the length to keep true time. Editor.]

PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

Give a general proof that the centre of gravity, or centroid, determines that point from which the sum of the distances to all other points of a given area is the minimum.

50. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Describe and compute the actual path traversed by the moon in July and August, 1896, taking into account the motion of the earth around the sun.

51. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to a city, and bought 156 horses and 177 mules for \$23631; he then traveled farther due north to A, and bought at same price 468 horses and 235 mules for \$52245; he then traveled from A due west 130 miles to B, and bought 120 cows; he then traveled due north to C, and bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 goats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 as much for goats as sheep; at D he turned and traveled in a straight line to the city, a distance equal to the sum of the entire distance he traveled due north from his home H; he sold all his stock at a profit of 20%. How far did he travel from his home H the entire trip around and back to the city? What was the cost of each head of stock, and what was the entire gain?

52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Pennsylvania.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes 1-4 of its circumference at the base.

NOTES.

NOTE ON ARTICLE IN AUGUST-SEPTEMBER NUMBER, VOL. III.

BY WARREN HOLDEN.

Referring to the demonstration on page 207 (current volume) without disputing the conclusion, allow me to submit the following considerations:

In Algebra, when zero is a factor in any term, the product is zero. Accordingly $0\times\infty=0$. In the course of the demonstration appears the expression $\frac{0\times1}{0}=\frac{0}{0}$, or the denominators being equal, $0\times1=0$. Would this result affect the conclusion in any way?

NOTE ON ELIMINATION.

BY J. C. CORBIN, PINE BLUFF, ARKANSAS.

The operation of elimination by addition and subtraction may often be shortened by the process and rule given below:

I. $5x+7y=43.$

$11x+9y=69.$

To eliminate y. $(9\times5-7\times11)x=9\times43-7\times69$. $\because x=3$.

To eliminate x. $(11\times7-5\times9)y=11\times43-5\times69$. $\because y=4$.

II. $21x+20y=165$.

$77x-30y=295$.

To eliminate x. $(3\times21+2\times77)y=3\times165+2\times295$. $\because y=5$.

To eliminate y. $(11\times20+3\times30)x=11\times165-3\times295$.

This is, substantially, the Determinant method ; but it is derived from the ordinary algebraic process by omitting all unessential work. The rule is : The difference (sum) of the products containing x (y) is equal to the difference (sum) of the numerical products.

EDITORIALS.

A few complete sets of Vol. I. and Vol. II. are still left. We will send Vol. I. to any address in the United States for $2., and Vol. II. for $2.50. Send in your order at once.

Prof. J. A. Calderhead, of Curry University, Pittsburg, Pennsylvania, sent in $3. as his subscription to the MONTHLY for 1896. We are very thankful for the material encouragement the friends of the MONTHLY are giving it.

A conference of the American Mathematical Society will convene in room 35 of Ryerson Physical Laboratory of the University of Chicago, at 10 o'clock, Thursday forenoon, December 31, 1896. It is expected that the conference will have three or four sessions and will adjourn on Friday, January 1, 1897. During the sessions of this conference some very important subjects will be discussed. Let every one interested in Mathematics attend this conference.

BOOKS AND PERIODICALS.

The Elements of Plane Geometry. By Charles A. Hobbs, A. M., Mathematical Master in the Volkmann School, Boston, Mass. 8vo. Cloth and Leather Back, 240 pages. Price, 75 cents. New York: A. Lovell & Co.

In this book the author has taken what seems to him to be a middle ground between the method of the students' following set demonstrations of a number of propositions and that of the students' producing all the argument in the course of a demonstration from original resources. There are 720 original propositions throughout the book besides many numerical exercises. The book is worthy the recognition of teachers. B. F. B.

Number and Its Algebra: A Syllabus of Lectures on the Theory of Number and Its Algebra Introductory to a Course in Algebra. By Arthur Lefevre, C. E., Instructor in Pure Mathematics, University of Texas. 8vo. Cloth, 230 pages. Boston: D. C. Heath & Co.

From only a cursory examination of this book we can say that it occupies a unique place in the literature of Mathematics. A careful reading of its contents by teachers will make the concept of numbers clear, and place their applications and the teaching of them on a solid foundation.
B. F, F.

A Primer of the Calculus. By E. Sherman Gould, Member of American Society of Civil Engineers. 16mo. Boards, 92 pages. Price, 50 cents. New York: D. Van Nostrand Co.

This little work is a development of the infinitesimal Calculus as far as the first differentials of algebraic functions of one independent variable and their corresponding integrals. Its size permits it to be carried about in the coat pocket and thus the self-taught may have at his command a work which he may read and study during his leisure.
B. F. F.

Elements of the Differential Calculus. By Edgar W. Bass, Professor of Mathematics in the United States Military Academy. 12mo. Cloth, 354 pages. New York: John Wiley & Sons.

The author says: "This text-book has been prepared for the use of the cadets of the United States Military Academy who begin the subject with a knowledge of the elements of Algebra, Geometry, and Trigonometry which ranges from fair to excellent. * * * * My experience leads me to the belief that the more rigorous and comprehensive method of infinitesimals is suitable only for a treatise and not for a text-book intended for beginners."

The author has, therefore, laid the foundation of his book on the methods of limits—the most accurate and simple of all the methods of presentation. One among the many commendable features of the book is the numerous, beautiful, and accurate diagrams used to aid in establishing the various principles upon which the Calculus is based. In this respect, it will appeal most favorably to the beginner. The book is one I most heartily recommend, and it is to be hoped that the author will follow it up by an equally good work on the Integral Calculus.
B. F. F.

List of Transitive Substitution Groups of Degree Twelve. By G. A. Miller, Ph. D., Göttingen, Germany. Extracted from The Quarterly Journal of Pure and Applied Mathematics, No. 111, 1896, pages 193—284.

Dr. Miller has given the subject of Substitution a great deal of study and he has written a number of articles on it. These various articles may be found in the leading Mathematical Journals of America and Europe. Those who are interested in this subject will find this article very helpful.
B. F. F.

The Criterion for Two-Term Prismoidal Formulas. By Dr. George Bruce Halsted. Pamphlet, 14 pages.

This interesting and valuable paper was presented to the Texas Academy of Science at its meeting, April 5, 1896. It contains many historical references and gives a pretty full history of the development of that interesting formula. Write to Dr. Halsted for a copy.
B. F. F.

Projective Groups of Perspective Collineations in the Plane Treated Synthetically. Pamphlet, 34 pages.

A dissertation presented to the Faculty of the University of Kansas by Arnold Emch to attain the degree of Doctor of Philosophy. B. F. F.

The Outlook Illustrated Monthly Magazine, Number for October. Price, 10 cents. The Outlook Co., 13 Astor Place, New York.

This number contains a full accouut of Princeton's 150th Anniversary, by Henry Van Dyke, with pictures; The Boys' Republic, by Washington Gladden, with twelve pictures; William Morris: A Poet's Workshop, by R. F. Zueblin, with five pictures; The Founder of the Y. M. C. A., by Lord Kinnaird, with nine pictures. B. F. F.

Popular Astronomy. Edited by W. W. Payne and H. C. Wilson, Goodsell Observatory of Carlton College, Northfield, Minnesota.

The November number contains the following: The Teaching of Descriptive Astronomy; Sketch of Astronomical Work at Munich; Biography of Prof. H. A. Newton, New York Evening Post; The Theory of Probability—An Historical Sketch; The Moon; The Constitution and Function of Gases; The Twilight; The Fixed Stars; The Planets and Constellations for October; Variable Stars. B. F. F.

Prace Matematyczen-Fizyczne. Wydawane. Przez S. Dicksteina, Warsaw, Russia.

The Mathematical Gazette. Edited by F. S. Macauley, St. Paul's School, West Kensington, W. London, England. Price, 3s. per year.

The Gazette aims at satisfying a want felt by many students for a Journal of Elementary Mathematics and is especially intended to be useful to teachers. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, \$1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

Errata in October Number.

Page 246, line 3, for "5^{n+1}" read 5^{n-1}.
Page 246, line 14, insert + before last term of (1).
Page 246, line 15, for "$4^{\overline{\frac{n-1}{2}}}.5$" read $4^{\frac{n-2}{2}}.5$.
Page 246, line 19, insert + before last term in (2).
Page 247, line 12, for "$4626x^3$" read $4626x^5$, and for "$\times$" read $+$.
Page 248, line 9, complete parenthesis after numerator of next to last term.
Page 250, problem 72 should read $2\sqrt{2}+\sqrt{3}/(4+\sqrt{6}-\sqrt{2})$.
Page 251, line 7 from bottom, for "$(-x)$" read $(-a)$.
Page 252, l. 20, read $R=[F(C^2-4AB)+AE^2+BD^2-CD^2]/(4AB-C^2)$.
Page 252, line 2 from bottom, reverse last mark of parenthesis after F.
Page 253, line 5, for "(Em^2-2k)" read $(Em^2-2k)y$.
Page 254, line 2, second = should be +.
Page 255, line 14, for "$n=$," etc. read u.
Page 255, line 18, for "(3)" read (2).
Page 256, line 1, in denominator, for "$\sqrt[3]{m\div n}$" read $\sqrt[3]{n\div m}$.

THE
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VOL. III. DECEMBER, 1896. No. 12.

LIE'S VIEWS ON SEVERAL IMPORTANT POINTS IN MODERN MATHEMATICS.

By G. A. MILLER, Ph. D., Göttingen, Germany.

It is generally admitted that America has contributed comparatively little towards the advancement of the science of mathematics. During the last twenty years there has been a rapidly increasing progress in this direction. Several European countries have also moved forward at a rapid rate during this period, so that our relative position is not improving as rapidly as might be desired.

The standard of general scholarship required for the higher degrees at our better institutions is comparatively high but the number of important discoveries does not yet correspond to this standard. In fact, the two are not apt to advance very far together, for the field of mathematics is so extensive that most are compelled to choose between a superficial acquaintance with the whole range of mathematical research and an exhaustive knowledge of only a few subjects.

In view of these facts it is natural that there should be many who strive to lead American mathematical talent to those newer regions which seem to offer the most fruitful fields of investigation. While there is a great difference of opinion with respect to these regions yet the most successful investigators are in the best possible position to judge in regard to them.

The view expressed by Klein during last year, in his address on *Arithmetizing Mathematics*, that Lie in Leipzig, Germany, and Poincaré in Paris, France, are the two most active mathematical investigators of the present day, is quite generally held. The following translation of a part of the introductory remarks of an article* published during last year by the former of these may therefore be

*Berichte der Koenigl. Saechs. Gesellschaft, 1895.

of considerable interest, as it contains the views of the author in regard to several important points in mathematics, especially in regard to the most important newer regions.

"In this century the concepts known as substitution and substitution group, transformation and transformation group, operation and operation group, invariant, differential invariant, and differential parameter, appear continually more clearly as the most important concepts of mathematics. While the curve as the representation of a function of a single variable has been the most important object of mathematical investigation for nearly two centuries from Descartes, while on the other hand, the concept of transformation first appeared in this century as an expedient in the study of curves and surfaces, there has gradually developed in the last decades a general theory of transformations whose elements are presented by the transformation itself while the series of transformations, in particular the transformation groups, constitute the object.

The general theory of transformations is a branch of analysis in the sense that it can be developed by purely analytic methods. It has however the material geometrical property that its operations are not only conceivable but directly intuitive to a large extent.

If we consider that the difference between the analytic and the synthetic methods exists in the fact that the synthesist reasons with concepts while the analyst operates with symbols, according to fixed rules, we may see an important property of the theory of transformations in this that its theorems can be developed in an elegant analytic as well as in a perspicuous even intuitively clear manner. It is due to this fact that the theory of transformations is considerably simpler than the theory of substitutions.

It should be added that different branches of mathematics have contributed to the development of the theory of transformations and that many parts of mathematics have already been considerably advanced by means of this theory.

The theory of differential equations is the most important branch of mathematics. Each department of physics presents problems which depend upon the integration of differential equations. In general, the theory of differential equations involves the road towards the explanation of all natural phenomena which require time. While this theory has an infinite practical value it has also a corresponding theoretic importance since it leads in a rational manner to the study of new important functions and classes of functions."

Gottingen, Germany, October 26, 1896.

NUMBER, COUNTING, MEASUREMENT.

By GEORGE BRUCE HALSTED, M. A. (Princeton), Ph. D. (Johns Hopkins), Professor of Mathematics in the University of Texas, Austin, Texas.

Counting is essentially prior to measuring, but also the primary number concept is essentially prior to counting and necessary to explain the meaning, cause and aim of counting. It is here maintained that integral number had not a metric origin, nor was metric in its original purpose ; that integral number did not involve the idea of ratio, that in fact it was enormously simpler than that very delicate concept, *ratio*. Number is primarily a quality of an artificial individual. The stress laid upon it, the importance attached to this quality comes first from the advantage of being able to identify one of these artificial individuals. By artificial is meant "of human make." The characteristic of these artificial individuals is that each, though made an individual, is conceived as consisting of other individuals.

The primitive function of number is to serve the purposes of identification. But again, counting, which consists in associating with each primitive individual in an artificial individual a distinct primitive individual in a familiar artificial individual, is thus itself essentially the identification, by a one-to-one correspondence, of an unfamiliar with a familiar thing. Thus primitive counting decides which of the familiar groups of fingers is to have its numeric quality attached to the unfamiliar group counted.

This primitive use of number in defining by identification is illustrated by an ordinary pack of playing cards, where the identification of King, Queen, and Knave is not more clearly qualitative and opposed to every mode of measurement than is the identification of ace, deuce, and tray ; and indeed that the King outvalues the Knave has more to do with measurement than the fact that the ace outvalues the tray.

Counting implies first a known series of groups, mental wholes each made up of distinct wholes ; secondly an unfamiliar mental whole ; thirdly the identification of the unfamiliar group by its one-to-one correspondence with a familiar group of the known series.

Absolutely no idea of a unit, of measurement, of amount, of value or even of equality is necessarily involved or indeed ordinarily used. One counts when one wishes to find out whether the same group of horses has been driven back at night that were taken out in the morning ; where counting is a process of identification which it would seem intentionally humorous or comical to try to connect fundamentally with any idea of a unit of reference or of some *value* to be ascertained, or of the setting off of a horse as a sample unit of value and then equating the total value to the number of such units. Such an *argumentum in circulo* may perhaps be funny, but it is neither fact nor mathematics. Mathematics afterwards defines numerical equality by means of one-to-one correspondence,

which is absolutely distinct and away apart from the idea of ratio. We may say with perfect certainty that there is no implicit presence of the ratio idea in primitive number.

From the contemplation of the primitive individual in relation to the artificial individual spring the related ideas "one" and "many." An individual thought of in contrast to "a many" as not-many gives the idea of "a one." A many composed of "a one" and another "one" is characterized as "two." A many composed of "a one" and the special many "a two" is characterized as "three." And so on ; at first absolutely without counting, in fact before the invention of that patent process of identification now called counting. For a considerable period of its early life every child uses a number system consisting of only three terms, *one*, *two*, *many*, and no counting. As datum may be taken a psychical continuum, and distinctness may be found the outcome of a process of differentiation ; but what may be spoken of as the physically originated primitive individuals, however complete in their distinctness, have no numeric suggestion or quality. The intuitive but creative apperception and synthesis of a manifold must precede its conscious analysis which alone gives number. It is only to conceptual unities that the numeric quality pertains. Such conceptual unities are of human make and in a sense are not in nature, while on the other hand, though the world we consciously perceive is out and out a mental phenomenon, yet the primitive individuals, distinct things, while forming part of the artificial unities, exist in another way, in that they are subsisting somehow in nature as well as in conscious perception.

In reference to these fundamental matters some strange blunders have been made of late by eminent philosophers and teachers, not mathematicians.

The number-picture of a group is a selective photograph of the group, which takes or represents only one quality of the group, but takes that all at once.

This picture process only applies primarily to those particular artificial wholes which may be called discrete aggregates. But the overwhelming importance of the number-picture, primarily as a means of identification, led, after centuries of its use, to a human invention as clearly a device of man for himself as is the telephone. This was a device for making a primitive individual thinkable as a recognizable and recoverable artificial individual of the kind having numeric quality. This recondite device is measurement. Measurement is an artifice for making a primitive individual conceivable as an artificial individual of the group kind, and so having a number picture. The height of a horse, by use af the unit "a hand," is thinkable as a discrete aggregate and so has a number-picture identifiable by comparison with the standard set of pictures, that is by counting, as say 16.

In Euclid's wonderful Fifth Book a ratio is never a number. Newton, with the purpose of taking in the so-called surds or irrationals of arithmetic and algebra, assumed a ratio to be a number. Any continuity in his number-system comes then from the continuity in the magnitude whose ratio to a chosen unit for

that magnitude is taken. He never gave any arithmetical or algebraic proof of the continuity of any number-system.

Austin, Texas.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from June-July Number.]

II. Proofs Resulting from Straight-line Properties of the Circle.

XV. Let ABC be $\triangle$ right-angled at C. With either extremity, as B, of the hypotenuse, as a center, and with a radius equal to the hypotenuse, describe a circle. Produce the legs of the $\triangle$ to chords. One of the chords, as DE, will be a diameter.

Then $AC.CL=DC.CE$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

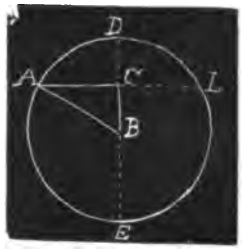


Fig. 11.

XVI. Let ABC be $\triangle$ right-angled at C. With either extremity, as B, of the hypotenuse, as the center, and with a radius equal to the adjacent leg, describe a circle. Produce the hypotenuse to a secant.

Then $\overline{AC}^2=AE.AD$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

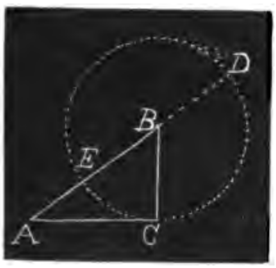


Fig. 12.

Note.—This method is given by Richardson in *Runkle's Mathematical Monthly*, No. II, 1859; also by Hoffmann, and, in a slightly different form, by Wipper, the latter stating that the proof is found in "Huberti Rudimenta Algebræ," Wurceb, 1792. It was known to the writers, however, independently of these sources.

XVII. Let ABC be a $\triangle$ right-angled at C.

Case I. When the two legs are unequal.

With C as a center, and with the shorter leg, as BC, as a radius, describe a circle. Produce AC to a secant. Draw CL perpendicular to AB.

Then $AD.AH=AE.AB$,

or $(b-a)(b+a)=c(c-2LB)$.

Fig. 13.

Substituting for LB any of its equivalents in terms of the sides of the given $\triangle$, and reducing, we get, $c^2=a^2+b^2$.

Case 2. When the two legs are equal.

We easily pass, by the usual method of the theory of limits, from Case 1 to Case 2.

XVIII. Same as in XVII, except that the circle is described with the longer leg, as AC, as a radius. Then, produce all the sides to chords.

Fig. 14.

Then $AB.BL=BE.BD$,

or $c.BL=(b+a)(b-a)$ ....(1).

Also, $AB : AH :: AC : AL$,

or. $c : 2b :: b : c+BL$,

whence, $c^2+c.BL=2b^2$ ....(2).

(1) in (2), $c^2=a^2+b^2$.

When the legs are equal, we pass from the case given as suggested in Case 2 of XVII.

XIX. Same as in XVII, except that both cases are treated alike, and the circle is described with a radius equal to the perpendicular from C to AB. Then produce the legs to secants, and draw CD.

Fig. 15.

Then $\overline{AD}^2=AH.AE=b^2-\overline{CD}^2$;

$\overline{BD}^2=a^2-\overline{CD}^2$;

also, $2AD.DB=2\overline{CD}^2$.

Adding, $c^2=a^2+b^2$.

XX. Let ABC be a $\triangle$ right-angled at C. Produce either leg, as AC, through C, making $CD=AC$. Join BD. Circumscribe a circle about $\triangle$ ABD, and produce BC to a diameter.

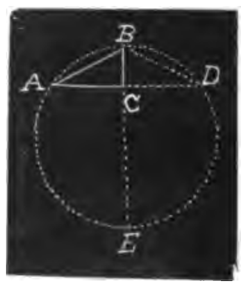


Fig. 16.

Then $\overline{BC}^2=AB.BD-AC.CD$,

or $a^2=c^2-b^2$.

$\therefore$ $c^2=a^2+b^2$.

XXI. Fig. 16.

$AB.BD=BE.BC$,

or $c^2=a^2+a.CE=a^2+b^2$.

NOTE.—The last two are special cases of familiar propositions, and are given by various writers.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

III. Solution by the PROPOSER.

Mr. Gruber's method is all right except the *assumption* that the length of the ditch increases as the price paid. The \$375 pays for 1890 hours' labor; at the same rate, \$500 would pay for 2520 hours' work. But there are only 2160 hours worked. Hence, the *efficiency* must be increased $\frac{1}{6}$. That is, the ditch will be $66\frac{2}{3}$ rods $\times\frac{7}{6}=77\frac{7}{9}$ rods long.

Or, in another light: Since 1890 hours' labor are worth \$375, 2160 hours' work, at same wages, are worth $\$428\frac{4}{7}$. But they get \$500, an increase of $\frac{1}{6}$ as before.

In this problem the *time* is limited—fixed—hence the only thing that can vary is the *efficiency* of the workmen. And it seems plain that it must increase as the *hourly* price increases—not as the *gross* price. Suppose

2 men in 1 day of 10 hours for \$20 dig x rods, and

3 men in 2 days of 10 hours for \$40 dig y rods. What is the ratio of y to x?

Can the *efficiency*, or productiveness, be found without considering the *hourly* wages?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds $m=10\%$ of water to the pure wine he buys, and then sells the mixture at a price $n=10\%$ greater than the cost price of the pure wine. What is his rate per cent. of profit?

Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let 100%=cost of the wine. Then 110% of $110\%=121\%$, the selling price of the mixture. Hence, $121\%-100\%=21\%$, the gain.

67. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth?

Solution by P. S. BERG, Larimore, North Dakota.

Since he received \$300 and a suit of clothes for a year, for one month he received \$25 and $\frac{1}{12}$ suit of clothes, and for five months he received \$125 and $\frac{5}{12}$ suit of clothes. He received \$60 and the clothes, hence \$60+suit of clothes= \$125+$\frac{5}{12}$ suit of clothes, or $\frac{7}{12}$ suit=\$65. Whence once suit=\$111$\frac{3}{7}$.

Also solved by *E. W. MORRELL* and *JAMES F. LAWRENCE*.

68. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $P=68921$, what was it $n=3$ years ago? At this rate of increase, what will the population be $n=3$ years hence?

Solution by P. S. BERG, Larimore, North Dakota.

Let 100%=what the population was 3 years ago. Then the population at present is $(100\%+2\frac{1}{2}\%)^3$. Hence $(100\%+2\frac{1}{2}\%)^3=68921$. Whence $100\%=$ 64000, the population 3 years ago. In 3 years hence the population will be $(100\%+2\frac{1}{2}\%)^3$ of 68921, or 74220.378765625.

69. Proposed by EDGAR M. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats, 13 Farmers; 3 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrets and Farmers. Required the number of men in the group.

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas.

Using obvious abbreviations, we can form the following table in which each small letter denotes a man:

Methodists.	Democrats.	Farmers.
a, b	a, b	a, b
c, d, e, f, g	h, i, j, k	h, i, j, k
l, m, n	l, m, n	r, s
	o, p, q	

Counting each letter once only, gives 19; 10 in the first column, 12 in the second column, and 13 in the third column.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Methodists.	Democrats.	Farmers.	Total.
3	3	0	3
0	4	4	4
5	0	5	5
2	2	2	2
10	9	11	14
0	3	2	5
10	12	13	19

∴ 19 men in the group.

Also solved by *E. W. MORRELL*, *JAMES F. LAWRENCE* and *P. S. BERG*.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

66. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2} \quad \text{.......(1)},$$

$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2} \quad \text{.......(2)}.$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x=r\cos\theta$, $y=r\sin\theta$. Then the equations become

$$a^2\cos\theta=2r^2\cos^2\theta-a^2 \quad \text{.......(1)},$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2 \quad \text{.......(2)}.$$

Eliminating r^2 from (1) and (2) we get

$$\frac{b^2(1+\sin\theta)}{1+\cos\theta}=a^2\tan^2\theta \quad \text{.......(3)}.$$

Now $\sin\theta=\dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$, $\cos\theta=\dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$. $\therefore$ $b^2(1+\tan\frac{1}{2}\theta)^2=2a^2\tan^2\theta$,

$$\text{or } b(1+\tan\tfrac{1}{2}\theta)=\pm\sqrt{2}\,a\tan\theta \quad \text{.......(4)}.$$

Let $z=\tan\frac{1}{2}\theta$; then (4) becomes

$$z^3+z^2-\left(1\mp\frac{2\sqrt{2}a}{b}\right)z=1 \quad \text{.......(5)}.$$

Let $u=z-\frac{1}{3}$; then (5) becomes

$$u^3-\tfrac{2}{3}\left(2\mp\frac{3\sqrt{2}a}{b}\right)u=\tfrac{2}{27}\left(8\pm\frac{9\sqrt{2}a}{b}\right) \quad \text{.......(6)}.$$

When a and b are known we can find u from (6), after which z and r and finally x and y become known.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

As in preceding solution,

$$a^2\cos\theta=2r^2\cos^2\theta-a^2 \quad \text{.......(3)},$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2 \quad \text{.......(4)}.$$

$\therefore\ 2r^2\cos^2\theta=a^2(1+\cos\theta)$......(5), and $2r^2\sin^2\theta=b^2(1+\sin\theta)$......(6).

Dividing (5) by (4), $\tan^2\theta=\dfrac{b^2}{a^2}\left(\dfrac{1+\sin\theta}{1+\cos\theta}\right)=\dfrac{b^2}{a^2}\left(\dfrac{\sec\theta+\tan\theta}{\sec\theta+1}\right)$......(7).

$\therefore\ a^2\tan^2\theta\sec\theta+a^2\tan^2\theta=b^2\sec\theta+b^2\tan\theta$........................(8).

$\therefore\ (a^2\tan^2\theta-b^2)\sec\theta=b^2\tan\theta-a^2\tan^2\theta$..........................(9).

Squaring (9) and substituting for $\sec^2\theta$ its value $1+\tan^2\theta$, performing operations indicated and arranging with reference to $\tan\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^4\theta+2a^2b^2\tan^3\theta-2a^2b^2t^2+b^4=0 \text{.........(10).}$$

Transposing the three middle terms and subtracting $2a^2b^2\tan^3\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^3\theta+b^4=2a^2b^2(\tan^4\theta-2\tan^3\theta+\tan^2\theta) \text{ ......(11).}$$

Extracting square root, $a^2\tan^3\theta-b^2=\pm ab(\tan^2\theta-\tan\theta)\sqrt{2}$........(12).

$$\text{Whence }\tan\theta=\frac{\pm d\sqrt{2}}{3}+\frac{1}{3a}\Big[\frac{b^2}{2}(9a\pm4b\sqrt{2})$$

$$+\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\Big]^{\frac{1}{3}}$$

$$+\frac{1}{3a}\Big[\frac{b^2}{2}(9a\pm4b\sqrt{2})-\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\Big]^{\frac{1}{3}}.$$

From equation (5), $x=a\sqrt{\dfrac{1+\cos\theta}{2}}=a\cos\frac{1}{2}\theta$, and from equation (6)

$$y=b\sqrt{\frac{1+\sin\theta}{2}}=b\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)=\frac{b}{\sqrt{2}}(\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta).$$

If $a=b$, from (12), $\tan\theta=1$ or $\dfrac{\pm\sqrt{2}-1}{2}\pm\frac{1}{2}(\pm2\sqrt{2}-1)^{\frac{1}{2}}$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Dividing (1) by (2) and putting $y=tx$, we obtain

$$x^2=\frac{a^2b^2(1-t)}{2t(a^2t-b^2)},\text{ and then } y^2=\frac{a^2b^2t(1-t)}{2(a^2t-b^2)}.$$

Substituting these in (1), we obtain finally the equation

$$t^6-\frac{2b^2}{a^2}t^4+\frac{2b^2}{a^2}t^3-\frac{2b^2}{a^2}t^2+\frac{b^4}{a^4}=0.$$

Solving this for numerical values of a and b, we get the values of x and y from the above expressions.

The same equation may be arrived at by putting $x=r\cos\theta$, $y=r\sin\theta$. The given equation then changes into $a^2\cos\theta=2r^2\cos^2\theta-a^2$; $b^2\sin\theta=2r^2\sin^2\theta-b^2$. Adding, we get $a^2\cos\theta+b^2\sin\theta=2r^2-(a^2+b^2)$, whence $r^2=\frac{1}{2}[a^2\cos\theta+b^2\sin\theta+a^2+b^2]$. Also, $r^2=\frac{1}{2}.\dfrac{a^2\cos\theta+a^2}{\cos^2\theta}$.

Equalizing, changing into the tangent function, the latter being denoted by t, we obtain the same equation as above.

IV. **Solution by H. C. WILKES, Skull Run, West Virginia.**

Putting $x^2+y^2=s^2$; then from (1), $a^2(x+s)=2sx^2$, and from (2), $b^2(y+s)=2sy^2$. Any rational value for s will give integral [?] fractional values for a^2 and b^2. Let $s=5$, $a^2=45/4$, and $b^2=160/9$; $s=13$, $a^2=325/9$, and $b^2=3744/25$; $s=17$, $a^2=2176/25$, and $b^2=3825/16$.

67. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.**

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* or *even*, [and not a multiple of 7].

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

$$\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{n\pi}{7}.$$

$$\sin\frac{4n\pi}{7}-\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{3n\pi}{7}.$$

$$\sin\frac{6n\pi}{7}-\sin\frac{4n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{5n\pi}{7}.$$

$$\therefore \cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{\sin\frac{6}{7}n\pi}{2\sin\frac{1}{7}n\pi}=\frac{\sin(n\pi-\frac{1}{7}n\pi)}{2\sin\frac{1}{7}n\pi}$$

$=-\frac{1}{2}\cos n\pi=\pm\frac{1}{2}$, according as n is odd or even.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

Employing the well-known formula

$$\sum_{n=1}^{n=n}\cos[a+(n-1)b]=\frac{\cos[a+\frac{1}{2}(n-1)b]\sin\frac{1}{2}nb}{\sin\frac{1}{2}b},$$

and putting $b=2a$, $n=3$, we have $\cos a+\cos3a+\cos5a=\dfrac{\cos3a\sin3a}{\sin a}=\frac12\dfrac{\sin6a}{\sin a}$.

But either $\frac12\dfrac{\sin6a}{\sin a}=\frac12\dfrac{\sin6a+\sin a}{\sin a}-\frac12=\dfrac{\sin\frac72a.\cos\frac52a}{\sin a}-\frac12$,

or, $\frac12\dfrac{\sin6a}{\sin a}=\frac12\dfrac{\sin6a-\sin a}{\sin a}+\frac12=\dfrac{\cos\frac72a.\sin\frac52a}{\sin a}+\frac12$.

Putting $a=\frac17n\pi$, we get in the former case $\dfrac{\sin\frac12n\pi\cos\frac5{14}n\pi}{\sin\frac17n\pi}-\frac12$, and in the latter $\dfrac{\cos\frac12n\pi\sin\frac5{14}n\pi}{\sin\frac17n\pi}+\frac12$. If n is *even*, $\sin\frac12n\pi=0$, if *odd*, $\cos\frac12n\pi=0$.

Q. E. D.

III. Solution by OTTO O. CLAYTON, A. B., Fowler, Indiana.

Unite 1st and 3rd terms of the left member ; then by factoring, we have,

$$(2\cos\tfrac27n\pi+1)\cos\tfrac37n\pi=\tfrac12 \text{ or } -\tfrac12.$$

Substituting for $(2\cos\frac27n\pi+1)$, we have $\dfrac{\sin\frac37n\pi\cos\frac37n\pi}{\sin\frac17n\pi}=\frac12$ or $-\frac12$, from which $\frac12\dfrac{\sin\frac67n\pi}{\sin\frac17n\pi}=\frac12.\dfrac{\sin-\frac17n\pi}{\sin\frac17n\pi}=\frac12$ or $-\frac12$.

This being an identical equation the problem is proved ; for ratio

$$\frac{\sin-\frac17n\pi}{\sin\frac17n\pi}=1 \text{ or } -1, \text{ according as } n \text{ is odd or even.}$$

IV. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington. Indiana.

The equation (1), $(\cos\theta+i\sin\theta)^7=-1$, *i. e.*, $\cos7\theta+i\sin7\theta=-1$, is clearly satisfied when θ has either of the following values : $\frac17\pi$, $\frac37\pi$, $\frac57\pi$, $\frac77\pi$, $\frac97\pi$, $\frac{11}7\pi$ and $\frac{13}7\pi$.

$\therefore$ (2), $(\cos n\theta+i\sin n\theta)^7=(-1)^n$ is satisfied by $\theta=\frac17\pi$, $\frac37\pi$, $\frac57\pi$, $\frac77\pi$, $\frac97\pi$, $\frac{11}7\pi$ or $\frac{13}7\pi$, or $n\theta=\frac17n\pi$, $\frac37n\pi$, $\frac57n\pi$, $\frac77n\pi$, $\frac97n\pi$, $\frac{11}7n\pi$ or $\frac{13}7n\pi$.

But (3), $(\cos n\theta+i\sin n\theta)^7=\cos^7n\theta+7i\cos^6n\theta\sin n\theta-21\cos^5n\theta\sin^2n\theta$ $-35i\cos^4n\theta\sin^3n\theta+35\cos^3n\theta\sin^4n\theta+21i\cos^2n\theta\sin^5n\theta-7\cos n\theta\sin^6n\theta-i\sin^7n\theta$ $=(-1)^n$.

$\therefore$ (4), $\cos^7n\theta-21\cos^5n\theta\sin^2n\theta+35\cos^3n\theta\sin^4n\theta-7\cos n\theta\sin^6n\theta=(-1)^n$.

Or (5), $64\cos^7n\theta-112\cos^5n\theta+56\cos^3n\theta-7\cos n\theta-(-1)^n=0$, of which

$\cos\frac{1}{7}n\pi$, $\cos\frac{3}{7}n\pi$, $\cos\frac{5}{7}n\pi$, $\cos\frac{7}{7}n\pi$, $\cos\frac{9}{7}n\pi$, $\cos\frac{11}{7}n\pi$ and $\cos\frac{13}{7}n\pi$ are the ratio.

Now $\cos\frac{7}{7}n\pi=\mp1$, according as n is *odd* or *even*, and $\cos\frac{13}{7}n\pi=\cos\frac{1}{7}n\pi$; $\cos\frac{9}{7}n\pi=\cos\frac{5}{7}n\pi$.

Hence we have (6), $(\cos n\theta\pm1)(64\cos^6 n\theta\mp64\cos^5 n\theta-48\cos^4 n\theta\pm48\cos^3 n\theta+8\cos^2 n\theta\mp\cos n\theta+1)=0$, according as n is *odd* or *even*.

$\therefore$ (7), $2(\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi)=\pm\frac{8}{8}=\pm1$. Or $\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi=\pm\frac{1}{2}$, according as n is *odd* or *even*.

We might deduce a number of equally interesting results, thus,

$$(\cos\tfrac{1}{7}n\pi.\cos\tfrac{3}{7}n\pi.\cos\tfrac{5}{7}n\pi)^2=\tfrac{1}{64}.$$

$\therefore$ $\cos\frac{1}{7}n\pi.\cos\frac{3}{7}n\pi.\cos\frac{5}{7}n\pi=\pm\frac{1}{8}$, when n is either *odd* or *even*, etc.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University of Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone if the angle at its vertex is less than a right angle.

II. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Assume axes of coördinates at right angles.

(a) The equation of the surface of a cone with axis of z as axis of cone, and origin at the vertex of cone is

$$x^2+y^2-z^2\tan^2\tfrac{1}{2}v=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where v=angle at vertex.

(b) The equation of a plane to same axes and origin as above, in terms of its direction cosines and perpendicular from origin is

$$lx+my+nz=\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminate z between (1) and (2), and then we have the conic

$$(n^2-l^2\alpha^2)x^2-2lmxy\alpha^2+y^2(n^2-m^2\alpha^2)+2pl\alpha^2x+2pm\alpha^2y-p^2\alpha^2=0\,..(3)$$

in which α is substituted for $\tan\frac{1}{2}v$.

In order that this conic may be an equilateral hyperbola, the angle between its asymptotes

$$(n^2-l^2\alpha^2)x^2-2lm\alpha^2xy+(n^2-m^2\alpha^2)y^2=0$$

must be a right angle, the condition for which is (for rectangular axes)

$$n^2-l^2\alpha^2+n^2-m^2\alpha^2=0, \text{ or } \alpha^2=2n^2/l^2+m^2 \;\ldots\ldots\ldots\ldots(4).$$

Now, in order that the plane above considered shall cut out the hyperbola, the angle whose direction cosine is n must be less than $\frac{1}{2}v$; that is to say, l and m must both be less than n. Hence, l^2+m^2 is necessarity less than n^2+n^2 or $2n^2$; or the fraction (4) is an improper fraction, whence $\alpha^2(=\tan^2\frac{1}{2}v)$ is greater than unity. This establishes that $\frac{1}{2}v$ is greater than 45°, and v is greater than 90°. Q. E. D.

III. Solution by GEORGE LILLEY, LL. D., Portland, Oregon.

Let ABF be a section of the cone made by the plane of the paper passing through its axis AM; $OPQNH$ any section of the cone made by a plane perpendicular to the plane ABF. Pass a plane through P at right angles to AM cutting the plane ABF in DE. Draw OL parallel to BF, and HK parallel to AF. Let $\angle MAB=\alpha$, $\angle AOH=\theta$, $AO=c$, $OH=x$, and $HP=y$.

$$HP^2=HD\times HE.\quad HE=\frac{x\sin\theta}{\cos\alpha},$$

$$DH=LK=2c\sin\alpha-\frac{x\sin(\theta+2\alpha)}{\cos\alpha}.$$

$$\therefore\ y^2=\frac{2c\sin\theta\sin\alpha}{\cos\alpha}x-\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}x^2.$$

The section represented by the equation is any hyperbola when $\theta+2\alpha$ is greater than 180°. Comparing the equation with $y^2=\frac{2b^2}{a}x+\frac{b^2}{a^2}x^2$, we have

$$\frac{2b^2}{a}=\frac{2c\sin\alpha\sin\theta}{\cos\alpha},\quad \frac{b^2}{a^2}=-\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}.$$

$$\therefore\ 2a=\frac{c\sin2\alpha}{\sin(\theta+2\alpha)},\quad b^2=\frac{c^2\sin^2\alpha}{\sin(\theta+2\alpha)}.$$

$e^2=\frac{a^2+b^2}{a^2}=\frac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{\cos^2\alpha}$, where e is the excentricity of the hyperbola. Or $1=\frac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{e^2\cos^2\alpha}$.

$e^2\cos^2\alpha$ must not be greater than unity. But $e^2=2$; therefore, $\cos^2\alpha$ must not be greater than $\frac{1}{2}$, and α must not be less than 45°. Hence, the angle at the vertex of the cone must not be less than a right angle; therefore, it is greater than a right angle.

It may, however, be equal to that angle.

Note on the angle between the asymptotes of the hyperbola.

Let ϕ=the angle between the asymptotes, and we have $\sec\frac{1}{2}\phi=e$, where e is the excentricity of the hyperbola.

$\sec^2\frac{1}{2}\phi\cos^2\alpha=e^2\cos^2\alpha$, or $\frac{\cos^2\alpha}{\cos^2\frac{1}{2}\phi}=e^2\cos^2\alpha$, but $e^2\cos^2\alpha$ must not be be greater than unity, see solution of problem 63. Hence, $\cos\frac{1}{2}\phi$ must not be less than $\cos\alpha$ and α must not be less than $\frac{1}{2}\phi$; or the angle at the vertex of the right circular cone, from which the hyperbola is cut, must not be less than the angle between the asymptotes. *It may, however, be equal to that angle.*

IV. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

If the axis of the cone be the axis z; h, the distance of the vertex from the origin, and θ the semi-angle at the vertex, the equation of the cone is

$$(x^2+y^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

The section of this cone made by a plane through the axis y is a conic section, and if the angle which the plane makes with xy be ϕ and the curve of intersection be referred to axes in its own plane, its equation is

$$y^2\tan^2(90^\circ-\theta)+x^2\cos^2\phi[\tan^2(90^\circ-\theta)-\tan^2\phi]+2hx\sin\phi-h^2=0.$$

If this is a rectangular hyperbola, then

$$\tan^2(90^\circ-\theta)=\cos^2\phi[\tan^2\phi-\tan^2(90^\circ-\theta)].$$

$\therefore\ \tan\phi=\pm\frac{\sqrt{2\sin(90^\circ-\theta)}}{\sqrt{\cos^2(90^\circ-\theta)}}$. But ϕ is real.

$\therefore\ 90^\circ-\theta<45^\circ$. $\therefore\ -\theta<-45^\circ$. $\therefore\ 2\theta>90^\circ$.

An hyperbola may also be cut from this cone by a plane parallel to the axis z, Its equation then is, if the cutting plane is $y=a$,

$$(x^2+a^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

If this be a rectangular hyperbola,

$$\tan^2(90^\circ-\theta)=1 \qquad \tan(90^\circ-\theta)=1 \qquad (-1 \text{ makes } \theta \text{ negative}).$$

$$\therefore\ 90^\circ-\theta=45^\circ. \quad \therefore\ \theta=45^\circ. \quad \therefore\ 2\theta=90^\circ.$$

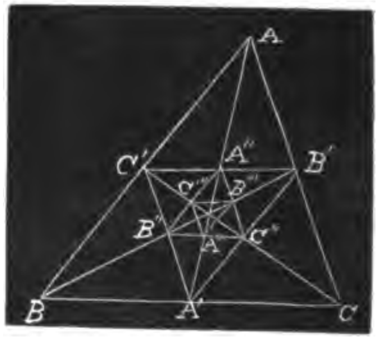
This problem was also solved in an excellent manner by *G. B. M. ZERR*.

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we have $\beta-\gamma=0$, $\gamma-\alpha=0$, $\alpha-\beta=0$ for the equations to AA', BB', CC' respectively.



$$\left(0,\ \frac{2\triangle}{b+c},\ \frac{2\triangle}{b+c}\right),\ \left(\frac{2\triangle}{a+c},\ 0,\ \frac{2\triangle}{a+c}\right),$$

$$\left(\frac{2\triangle}{a+b},\ \frac{2\triangle}{a+b},\ 0\right)$$

are the coordinates of A', B', C' respectively.

$\therefore\ \alpha+\beta-\gamma=0$, $\alpha+\gamma-\beta=0$, $\beta+\gamma-\alpha=0$ are the equations to $A'B'$, $A'C'$, $B'C'$ respectively.

$$\left(\frac{4\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c}\right),\ \left(\frac{2\triangle}{a+2b+c},\ \frac{4\triangle}{a+2b+c},\ \frac{2\triangle}{a+2b+c}\right),$$

$$\left(\frac{2\triangle}{a+b+2c},\ \frac{2\triangle}{a+b+2c},\ \frac{4\triangle}{a+b+2c}\right)$$

are the coordinates of A'', B'', C'' respectively.

$\therefore\ \alpha+\beta-3\gamma=0$, $\alpha+\gamma-3\beta=0$, $\beta+\gamma-3\alpha=0$ are the equations to $A''C''$, $A''C''$, $B''C''$ respectively.

$$\left(\frac{4\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c}\right),\ \left(\frac{6\triangle}{3a+2b+3c},\ \frac{4\triangle}{3a+2b+3c},\right.$$

$$\frac{6\triangle}{3a+2b+3c}\Big),\ \Big(\frac{6\triangle}{3a+3b+2c},\ \frac{6\triangle}{3a+3b+2c},\ \frac{4\triangle}{3a+3b+2c}\Big)$$

are the coordinates of A''', B''', C''' respectively.

$\therefore$ $3\alpha+3\beta-5\gamma=0$, $3\alpha+3\gamma-5\beta=0$, $3\beta+3\gamma-5\alpha=0$ are the equations to $A'''B'''$, $A'''C'''$, $B'''C'''$ respectively.

$$\Big(\frac{12\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c}\Big),\ \Big(\frac{10\triangle}{5a+6b+5c},\ \frac{12\triangle}{5a+6b+5c},$$

$$\frac{10\triangle}{5a+6b+5c}\Big),\ \Big(\frac{10\triangle}{5a+5b+6c},\ \frac{10\triangle}{5a+5b+6c},\ \frac{12\triangle}{5a+5b+6c}\Big)$$

are the coordinates of A'''', B'''', C'''' respectively.

$\therefore$ $5\alpha+5\beta-11\gamma=0$, $5\alpha+5\gamma-11\beta=0$, $5\beta+5\gamma-11\alpha=0$ are the equations to $A''''B''''$, $A''''C''''$, $B''''C''''$ respectively.

In what follows, the upper signs are used for m odd, and the lower for m even. The mth term of the series 1, 3, 5, 11, 21, 43, 85, etc., is $\frac{1}{3}(2^m\pm1)$.

$$\therefore\ \Big(\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},$$

$$\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)}\Big),\ \Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)}\Big),$$

$$\Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},\ \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)}\Big)$$

are the coordinates of A^m, B^m, C^m respectively.

$\therefore$ $\frac{1}{3}(2^m\pm1)(\alpha+\beta)-\frac{1}{3}(2^{m+1}\mp1)\gamma=0$, $\frac{1}{3}(2^m\pm1)(\alpha+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\beta=0$, $\frac{1}{3}(2^m\pm1)(\beta+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\alpha=0$, (1, 2, 3) are the equations to A^mB^m, A^mC^m, B^mC^m respectively.

From (1) and (2), (1) and (3), (2) and (3) respectively, we get

$$\tan A^m=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin A+2^m(2^m\pm1)(\sin B+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos A-2(2^m\pm1)(2^{m-1}\mp1)(\cos B+\cos C)}$$

$$\tan B^{m}=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin B+2^{m}(2^{m}\pm1)(\sin A+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^{m}+1)\cos B-2(2^{m}\pm1)(2^{m-1}\mp1)(\cos A+\cos C)}$$

$$\tan C^{m}=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin C+2^{m}(2^{m}\pm1)(\sin A+\sin B)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^{m}+1)\cos C-2(2^{m}\pm1)(2^{m-1}\mp1)(\cos A+\cos B)}$$

Let A=area of $A^{(m)}B^{(m)}C^{(m)}$, p=perpendicular from $C^{(m)}$ on $A^{(m)}B^{(m)}$.

$$\therefore\ A=[27abc\triangle.2^{m}]\div[\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}$$
$$\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}]$$

$$p=[9\triangle.2^{m+1}]\div[\{(2^{m}\pm1)(a+b)+2(2^{m-1}\mp1)c\}$$
$$\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos B)-2(2^{m}\pm1)^{2}\cos C}].$$

But $A=\frac{1}{2}pA^{(m)}B^{(m)}$. $\therefore\ A^{(m)}B^{(m)}=2A/p$.

$\therefore\ A^{(m)}B^{(m)}$

$$=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos B)-2(2^{m}\pm1)^{2}\cos C}}{\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}}$$

$$A^{(m)}C^{(m)}=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos A+\cos C)-2(2^{m}\pm1)^{2}\cos B}}{\{2(2^{m-1}\mp1)a+(2^{m}\pm1)(b+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}}$$

$$B^{(m)}C^{(m)}=\frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^{m}\pm1)(2^{m+1}\mp1)(\cos B+\cos B)-2(2^{m}\pm1)^{2}\cos A}}{\{2(2^{m-1}\mp1)b+(2^{m}\pm1)(a+c)\}\{2(2^{m-1}\mp1)c+(2^{m}\pm1)(a+b)\}}$$

All principles necessary to understand the above solution will be found in the chapter on "Trilinear Coordinates" in Todhunter's Conic Sections.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. Proposed by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$, and show that $x=y\log(xy)$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; and P. S. BERG, Larimore, North Dakota.

Clearing of fractions we obtain after transposing two terms

$$y(xdy+ydx)=x(ydx-xdy).$$

Dividing by y^2, we have

$$xdy+ydx=xy.\frac{ydx-xdy}{y^2}, \text{ or, } d(xy)=xy.d\left(\frac{x}{y}\right). \quad \therefore \frac{d(xy)}{xy}=d\left(\frac{x}{y}\right).$$

Integrating, $\log(axy)=(x)\div(y)$, whence $x=y\log(axy)$.

The result given is not general enough, the constant having been left out of consideration.

II. Solution by W. W. LANDIS, A. M., Department of Mathematics and Astronomy in Dickinson College, Carlisle, Pennsylvania; F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey; J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; and A. H. HOLMES, Brunswick, Maine.

Let $y=vx$, then the equation becomes

$$v+\frac{xdv}{dx}=\frac{x(1-v)}{1+v}, \text{ or } 2v^2+\frac{x(1+v)dv}{dx}=0. \quad \therefore \frac{1+v}{2v^2}dv+\frac{dx}{x}=0.$$

The variables are separable, whence

$$\frac{dv}{2v^2}+\frac{dv}{2v}+\frac{dx}{x}=0. \quad \text{Integrating, } \frac{1}{2v}=\tfrac{1}{2}\log v+\log x.$$

$\therefore$ $1/v=\log v+\log x^2=\log(vx^2)$.

$\therefore$ $x/y=\log(xy)$, or $x=y\log(xy)$, when no constant is added, or $x=y\log(xy)+Cy$ where C is an arbitrary constant.

III. Solution by M. C. STEVENS, M. A., Mathematical Department, Purdue University, Lafayette, Indiana; HENRY HEATON, M. Sc., Atlantic, Iowa; JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington, Indiana; and J. C. GREGG, A. M., Brazil, Indiana.

Put $y=vx$ and we have

$$v+x\frac{dv}{dx}=\frac{vx^2-v^2x^2}{x^2+vx^2}=\frac{v-v^2}{1+v}=v-\frac{2v^2}{1+v}.\ \text{ Whence, } \frac{1+v}{v^2}dv+\frac{2dx}{x}=0.$$

Integrating, $-1/v+\log(vx^2)+C=0$, or $x/y=\log(xy)+C$, and $x=y\log(xy)+Cy$.

The C should not be omitted unless the conditions of the question giving rise to the equation are such as to make it zero.

IV. Solution by H. C. WHITTAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Let $y=x^pv^q$ and substitute in the given equation and we obtain

$$\frac{dv}{dx}=\frac{(1-p)v-(1+p)x^{p-1}v^{q+1}}{q^x(1+x^{p-1}v^q}.$$

This will reduce to a simple form if we take $p=1$ and $q=-1$, giving

$$\frac{dv}{dx}=\frac{2}{x(1+v^{-1})},\ \text{or } dv(1+v^{-1})=2x^{-1}dx.$$

$v+\log v=\log x^2$; $x/y+\log(x/y)=\log x^2$.
$x/y=\log x^2-\log(x/y)=\log(xy)$, whence $x=y\log(xy)$.

54. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite side of the square are semi-circles, however. Find surface, volume, and center of gravity of each.

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

The length of a side of a parallel section distant x from the base is $(a^2-4x^2)^{\frac{1}{2}}$. If dx be the distance between two parallel sections, the distance between two corresponding sides is $adx/(a^2-4x^2)^{\frac{1}{2}}$. Hence the surface

$$S=4\int_0^{\frac{1}{2}a}adx=2a^2\ ;\ \text{the volume } V=\int_0^{\frac{1}{2}a}(a^2-4x^2)dx=\tfrac{1}{3}a^3\ ;$$

the distance of the center of gravity of the surface from the base is

$$\frac{1}{2a^2}\int_0^{\frac{1}{2}a}axdx=\tfrac{1}{4}a\ ;$$

and the distance of the center of gravity of the volume from the base is

$$\frac{3}{a^3}\int_0^{\frac{1}{2}a} x(a^2-4x^2)dx=\tfrac{3}{16}a.$$

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the State A. M. College, College Station, Texas.

Take the intersection of the planes of the circular sections as the axis of z, the origin being in the center of the base. Then since the radius of each circle is $\frac{1}{2}a$ we shall have for the projection of one fourth of the elementary area intercepted between two planes parallel to the base, and distant dz from each other, upon the plane of one of the circles,

$$dS.\cos\theta=\sqrt{\tfrac{1}{4}a^2-z^2}.dz,$$

where θ is the angle made by this elementary area with the plane of projection.

But $\cos\theta=\dfrac{\sqrt{\frac{1}{4}a^2-z^2}}{\frac{1}{2}a}$, and the whole surface is $S=4\displaystyle\int_0^{\frac{1}{2}a} adz=2a^2$....(1).

The center of gravity of S is distant from the base

$$z_1=\frac{4\int_0^{\frac{1}{2}a} azdz}{2a^2}=\tfrac{1}{4}a\text{......................}(2).$$

For the volume, taking planes parallel to the base,

$$V=\int_0^{\frac{1}{2}a} 2\sqrt{\tfrac{1}{4}a^2-z^2}.2\sqrt{\tfrac{1}{4}a^2-z^2}.dz=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=a^3/3\text{.........}(3),$$

and its center of gravity above the base is

$$z_2=\frac{\int_0^{\frac{1}{2}a}(a^2-4z^2)zdz}{\frac{1}{3}a^3}=\tfrac{3}{16}a\text{..... ....}(4).$$

The figure will be a cloistered arch formed by the intersection of two right semi-cylinders.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x^2+z^2=\frac{1}{4}a^2$......(1), $y^2+z^2=\frac{1}{4}a^2$......(2) be the equations to the cylinders which form the groin. From (1) $dz/dx=-x/z$, $dz/dy=0$.

$$S=\int\int\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy=8\int_0^{\frac{1}{2}a}\int_0^x\sqrt{1+\frac{x^2}{z^2}}\,dxdy$$

$$=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{z}=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{\sqrt{\frac{1}{4}a^2-x^2}}=4a\int_0^{\frac{1}{2}a}\frac{xdx}{\sqrt{\frac{1}{4}a^2-x^2}}=2a^2.$$

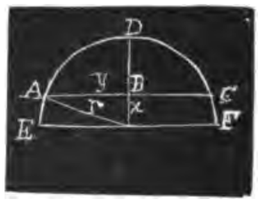
$$V=\int\int\int dzdxdy=4\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}dzdxdy=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=\tfrac{1}{3}a^3.$$

$$\text{Center of graving of surface}=\frac{\int\int zdS}{\int\int dS}=\tfrac{1}{2}a\int_0^{\frac{1}{2}a}\int_0^x dxdy=\tfrac{1}{4}a.$$

$$\text{Center of gravity of volume}=\frac{\int\int\int zdzdxdy}{\int\int\int dzdxdy}=\frac{3}{a^3}\int_0^{\frac{1}{2}a}z(a^2-4z^2)dz=\tfrac{3}{16}a.$$

IV. **Solution by J. C. GREGG, A. M., Brazil, Indiana.**

Let the given figure represent a section of the solid through the middle point of two opposite sides of the base. We have $r=a/2$, and the equation of the circle EDF is $x^2+y^2=r^2 \ldots\ldots(1)$, and $AC^2=(2y)^2=4(r^2-x^2)=A_x=$ a section parallel to the base, and for the volume



$$V=4\int_0^r(r^2-x^2)dx=\tfrac{8}{3}r^3=\tfrac{1}{3}a^3.$$

The surface may be considered to be generated by the sides of a section parallel to the base, and we have for the surface,

$$S=4\int 2yds=4\int_0^r 2\sqrt{r^2-x^2}.\frac{rdx}{\sqrt{r^2-x^2}},\ =8r\int_0^r dx=8r^2=2a^2.$$

For the center of gravity of the volume,

$$\bar{x}=\frac{\int x(A_x)dx}{V},\ =\frac{4\int_0^r x(r^2-x^2)dx}{\frac{8}{3}r^3},\ =\frac{3}{2r^3}\int_0^r x(r^2-x^2)dx=\tfrac{3}{8}r=\tfrac{3}{16}a.$$

For the center of gravity of the *curved* surface we have,

$$\bar{x}=\frac{4\int 2xyds}{S},\ =\frac{8r\int_0^r xdx}{8r^2}=\frac{1}{r}\int_0^r xdx,\ =\tfrac{1}{2}r,\ =\tfrac{1}{4}a.$$

For the center of gravity of the *whole* surface, since the curved surface is *twice* that of the base we have, $\bar{x}=\tfrac{2}{3}.\tfrac{1}{4}a=\tfrac{1}{6}a$.

Also solved by *H. C. WHITAKER*, *C. W. M. BLACK*, and the *PROPOSER*.

Professors Black and Scheffer used "side$=2a$" as in Problem 47, instead of *side*$=a$, and hence their results did not agree with those in the published solutions. The results obtained were : Volume$=8a^3/3$, surface$=8a^2$, center of gravity of volume$=3a/8$, and center of gravity of surface$=\tfrac{1}{3}a$. See problem 42 for two additional SOLUTIONS for surface and volume. EDITOR.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A prolate spheroid of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

The general equations of motion are :

$$\left.\begin{array}{l} A\omega_x-(\Sigma mxy)\omega_y-(\Sigma mxz)\omega_z=L \\ B\omega_y-(\Sigma myz)\omega_z-(\Sigma myx)\omega_x=M \\ C\omega_z-(\Sigma mzx)\omega_x-(\Sigma mzy)\omega_y=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the ellipsoid with focus as origin is $a^2y^2+a^2z^2+b^2x^2=2aeb^2x+b^4$. Now $\Sigma mxy=\Sigma mxz=\Sigma myz=0$. $\therefore$ (1) reduce to

$$\left.\begin{array}{l} A\omega_x=L \\ B\omega_y=M \\ C\omega_z=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let $2aeb^2x+b^4-b^2x^2=a^2c^2$. Then

$$A=\Sigma m(y^2+z^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(y^2+z^2)dxdydz$$

$$=\frac{4\mu}{3}\int_{-a(1-e)}^{a(1+e)}\int_{0}^{c}\{3y^2\sqrt{c^2-y^2}+(c^2-y^2)\sqrt{c^2-y^2}\}dxdy$$

$$=\frac{\pi\mu}{2a^4}\int_{-a(1-e)}^{a(1+e)}(2aeb^2x+b^4-b^2x^2)^2dx=\tfrac{8}{15}\mu\pi ab^4.$$

$$B=C=\Sigma m(x^2+y^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(x^2+y^2)dxdydz$$

$$=4\mu\int_{-a(1-e)}^{a(1+e)}\int_{0}^{c}(x^2+y^2)\sqrt{c^2-y^2}\,dxdy$$

$$=\frac{\mu\pi}{4}\int_{-a(1-e)}^{a(1+e)}(4c^2x^2+c^4)dx=\tfrac{8}{15}\mu\pi a^3b^2(1+2e^2).$$

Let the blow$=P$ be struck perpendicular to the plane (xy), then the moments of the impulsive forces about the axes are $L=Pb$, $M=Pae$, $N=0$.

These in (2) give

$$\left.\begin{array}{l}\tfrac{8}{15}\mu\pi ab^4\,\omega_x=Pb\\ \tfrac{8}{15}\mu\pi a^3b^2(1+2e^2)\omega_y=Pae\\ \omega_z=0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

$$\therefore\frac{\omega_y}{\omega_x}=\frac{e^2}{1+2e^2}\cdot\frac{b}{ae}.$$

Let F be the focus, O the center of the ellipsoid. Then on the minor axis in the plane (xy), take $OE=\frac{e^2}{1+2e^2}.b$, then will FE be the axis required.

Let $a=5$, $b=4$. $\therefore e=\frac{3}{5}$. $\therefore OE=\frac{9}{43}b=\frac{36}{43}$. The resultant angular velocity will be

$$\frac{15P}{8\mu\pi a^2b^3e}.OF=\frac{P}{512\mu\pi}.OF=\frac{3P\sqrt{1993}}{22016\mu\pi},\text{ when } a=5,\ b=4.$$

39. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Supposing the initial inclination of the swing to the vertical is α and that the person always crouches when in the highest position and stands up when in the lowest, his center of gravity moving through a distance b measured from lower part of swing upward, find how much the arc is increased after n complete vibrations.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Let RS be the path of the center of gravity from extreme position to vertical, TU the path from vertical to other extreme position.

$OP=l$, $RP=k$, (say), $TS=b$; $\angle ROS=a$, $\angle TOU=\alpha_1$, etc.

The energy acquired by swing in passing from R to S is $(l-k)(1-\cos\alpha)W$.

When it has passed to V the energy is $(l-k-b)(1-\cos\alpha_1)W$.

By conservation of energy

$$(l-k-b)(1-\cos\alpha_1)W=(l-k)(1-\cos\alpha)W.$$

Whence $1-\cos\alpha_1=\dfrac{l-k}{l-k-b}(1-\cos\alpha)$.

In passing back to original position

$$1-\cos\alpha_2=\frac{l-k}{l-k-b}(1-\cos\alpha_1)=\left(\frac{l-k}{l-k-b}\right)^2(1-\cos\alpha).$$

For two complete vibrations $1-\cos\alpha_4=\left(\dfrac{l-k}{l-k-b}\right)^4(1-\cos\alpha)$.

In like manner for n complete vibrations

$$1-\cos\alpha_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha) \text{ or } \sin(\tfrac{1}{2}\alpha_{2n})=\left(\frac{l-k}{l-k-b}\right)^{n}\sin(\tfrac{1}{2}\alpha),$$

which enables us to compute the increase in amplitude.

II. Solution by the PROPOSER.

Let O be the point of suspension of the swing, S the position of the center of gravity of the man when crouching, and T its position when the man is standing, and Q the lower end of the swing.

Let $OQ=l$, $SQ=k$, the distance from lower end of swing to the center of

gravity of the man when he is crouching, $ST=b$, $\angle QOP=\alpha$, and $\angle QOV=\theta_0$.

Then the velocity, v, of the man at the point $Q,=\sqrt{2gl(l-k)(1-\cos\alpha)}$, *Bowser's Analytical Mechanics*, page 350, Article 192. Hence, his energy due to his weight $=\dfrac{W}{2g}v^2=W(l-k)(1-\cos\alpha)$. This energy will carry him to V and equals $W(l-k-b)(1-\cos\theta_0)$, since the man rises at the point Q. Since he crouches at the point V, his energy at the point Q on his return is $W(l-k)(1-\cos\theta_0)$. This energy will carry him to a point to the left of S, and the energy expended will be $W(l-k-b)(1-\cos\theta_1)$, where θ_1 is the angle between the vertical and the swing.

According to the principle of the conservation of energy, we have,

$$W(l-k)(1-\cos\alpha)=W(l-k-b)(1-\cos\theta_1)\ldots\ldots\ldots\ldots(1),$$

$$W(l-k)(1-\cos\theta_0)=W(l-k-b)(1-\cos\theta_2)\ldots\ldots\ldots\ldots(2),$$

$$W(l-k)(1-\cos\theta_1)=W(l-k-b)(1-\cos\theta_3)\ldots\ldots\ldots\ldots(3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$W(l-k)(1-\cos\theta_{2n-1})=W(l-k-b)(1-\cos\theta_{2n-1})\ldots\ldots(2n).$$

Multiplying these equations together member for member, and solving for $1-\cos\theta_{2n}$, we have,

$$1-\cos\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha),\text{ or }\sin^2(\tfrac{1}{2}\theta_{2n})=\left(\frac{l-k}{l-k-b}\right)^{2n}(\sin^2\tfrac{1}{2}\alpha).$$

Whence, $\sin\frac{1}{2}\theta_{2n}=\left(\dfrac{l-k}{l-k-b}\right)^n\sin\frac{1}{2}\alpha$, or $\theta_{2n}=2\sin^{-1}\left[\left(\dfrac{l-k}{l-k-b}\right)\sin\frac{1}{2}\alpha\right]$.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, Chicago, Illinois.

Required all the parallelograms whose sides a, b and diagonals c, d, are rational.

II. Solution by W. F. KING, Ottowa, Canada.

The conditions are, $2(a^2+b^2)=c^2+d^2$, with the condition that a, b, c and a, b, d shall be capable of forming triangles (sum of any two sides greater than the third side). That is, if we suppose $a>b$, $c>d$, $a+b$ must be greater than c, and $a-b<d$. These two conditions again are the same, for if $a+b>c$ and $2(a^2+b^2)=c^2+d^2$, $2(a^2+b^2)-(a+b)^2<d^2$ or $a-b<d$.

Let us suppose the numbers involved to be integers. We have $c^2+d^2=2(a^2+b^2)=(a+b)^2+(a-b)^2$. If $c=a+b$, $d=a-b$, the parallelogram vanishes, the angle opposite to the diagonal c becoming 180°. But if not, we have a number c^2+d^2 which is resolvable into the sum of two squares in another way. Hence, as may easily be proved, a^2+b^2 is resolvable into factors, which, by a theorem in the Theory of Numbers, are (whether prime or composite) each expressible as the sum of two squares. Also every prime number of the form $4n+1$ is expressible as the sum of two squares, and every number which is the sum of two squares is the product of prime factors of the form $4n+1$. (The only even prime $2=1^2+1^2$ or a power thereof may also be a factor, which case will be considered further on).

Hence we have a rule to find a and b so as to make c and d rational.

Form a^2+b^2 by multiplying together two or more of the various prime numbers of the form $4n+1$, such as 5, 13, 17, 29, etc.

The product may be expressed in two ways at least as the sum of two squares. Thus we shall have $f^2+g^2=h^2+k^2$.

$\therefore$ $2(f^2+g^2)=(h^2+k)^2+(h-k)^2$, which gives a solution by putting $f=a$, $g=b$, $h+k=c$, $h-k=d$, provided that, (following the condition for a possible triangle) $f-g<h-k$.

If $f-g>h-k$, we must take h and k for a and b; $f+g$ and $f-g$ for c and d. Then $2(h^2+k^2)=(f+g)^2+(f-g)^2$.

That is, of the two equal sums into which the product has been resolved, take that for a^2+b^2 which has the less difference between its components.

For example, multiply 5 by $13=65$.

$65=8^2+1^2=7^2+4^2$ and $7-4=3<8-1$. Hence $a=7$, $b=4$, $h=8$, $k=1$, and $2(7^2+4^2)=(8+1)^2+(8-1)^2=9^2+7^2$.

$\therefore$ $c=9$, $d=7$. And 7, 4, 9; 7, 4, 7 are possible sides for triangles. The components of the product can readily be found from the components of the prime factors, thus:

Let $N=(p^2+q^2)(r^2+s^2)=p^2r^2+2pqrs+q^2s^2+q^2r^2-2pqrs+p^2s^2=p^2r^2-2pqrs+q^2s^2+q^2r^2+2pqrs+p^2s^2=(pr+qs)^2(qr-ps)^2=(pr-qs)^2+(qr+ps)^2$.

For example, to resolve 65 into the sum of two squares.

$65=13\times5=(3^2+2^2)(2^2+1^2)$. Here $pr+qs=3\times2+2\times1=8$.

$qr-ps=2\times2-3\times1=1$. $pr-qs=3\times2-2\times1=4$,

$qr+ps=2\times2+3\times1=7$. $\therefore$ $65=8^2+1^2=4^2+7^2$.

A third factor, $r_1^2+s_1^2$, can be introduced by putting $pr+qs=p_1$, $qr-ps=q_1$, and multiplying out $(p_1^2+q_1^2)(r_1^2+s_1^2)$ as before.

Observe that this gives two forms for the product, and two more would be got by putting $pr-qs=p_1$, $qr+ps=q_1$, so that with three factors there will be four forms for the product. These forms may be taken any two together giving $4.3/1.2=6$ solutions for c and d.

In the preceding it has been assumed that a and b have no common factor. If they have one (which may be any number whatever) the preceding investigation will still hold. But such factors of the common measure of a and b as are not primes of the form $4n+1$ will re-appear as common factors of c and d. It is to be noted that if $a^2+b^2=2\times$ a single prime factor a^2+b^2 can be expressed as the sum of two squares in one way only, viz: $a^2+b^2=2(p^2+q^2)=(p+q)^2+(p-q)^2$. For the factors of a^2+b^2 are p^2+q^2 and r^2+s^2 where $r=s=1$, and in multiplying $(p^2+q^2)(r^2+s^2)$ the two expressions $(qr+ps)^2+(pr-qs)^2$ and $(pr+qs)^2+(qr-ps)^2$ become identical when $r=s=1$, and these two cannot be equated to a third and different sum of two squares without factoring p^2+q^2, which is by supposition a prime. Hence when $\frac{1}{2}(a^2+b^2)$ is a prime the solution fails, for we get $2(a^2+b^2)=4(p^2+q^2)=(2p)^2+(2q)^2=(a+b)^2+(a-b)^2$, which does not give a parallelogram. So also when a^2+b^2 is a product of a prime by any odd power of 2. An even power of 2 may however be used, for example,

$$a^2+b^2=260=2^2\times5\times13=2^2(3^2+2^2)(2^2+1^2)=(6^2+4^2)(2^2+1^2)=16^2+2^2$$
$$=14^2+8^2 \text{ and } 2(a^2+b^2)=2(14^2+8^2)=2(16^2+2^2)=18^2+14^2=c^2+d^2.$$

The above discussion made on the assumption that a, b, c, d are integers, is readily extended to give solutions in rational fractions.

Thus $1885=5\times13\times29=42^2+11^2=34^2+27^2$.

$\therefore\ 7^2+(\frac{11}{6})^2=(\frac{17}{3})^2+(\frac{9}{2})^2$ and $2\{(\frac{17}{3})^2+(\frac{9}{2})^2\}=(\frac{53}{6})^2+(\frac{31}{6})^2$.

Note on solution of problem 37, page 151. The failure to give the least values in my solution was due to solving $x_1^2-40y_1^2=1$; by continued fractions, we obtain positive integral values for x and y, but y does not enter into the required values directly; hence may be fractional. This point was overlooked.

To obtain all these values, let $x=(x_2)\div(z_2)$, $y=(y_2)\div(z_2)$, and then $x_2^2-z_2^2=40y_2^2=10y_3^2$. $(x_2+z_2)(x_2-z_2)=10y_3^2$. Let $y_3^2=p^2q^2$ and $10=$any two factors.

$$\left.\begin{array}{l} x_2+z_2=p^2 \text{ or } 2p^2 \\ x_2-z_2=10q^2 \text{ or } 5q^2 \end{array}\right\}$$

Add and subtract, then

$$\left.\begin{array}{l} x_2=p^2+10q^2 \text{ or } 2p^2+5q^2 \\ z_2=\mp p^2\pm10q^2 \text{ or } \mp2p^2\pm5q^2 \\ y_3=2pq \end{array}\right\}$$

p and q taken at pleasure will give an infinite number of values, integral and fractional. Mr. Gruber's list is correct.

A. H. Bell.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

Note on Problem 33. By Henry Heaton, Atlantic, Iowa.

The result obtained in the published solution of this problem cannot be correct.

The area of the regular pentagon is $3.6327a^2$. The area of each of the infinite number of regular polygons whose apothem is a and number of sides greater than five, is less than $3.6327a^2$, while that of only two, the square and triangle, is greater. Hence the average area of all regular polygons with apothem a is less than $3.6327a^2$. Hence the result obtained in the published solution ($3.8693a^2$) is too large.

In a similar manner it may be shown that any result larger than $a^2\pi$ is too large, while it is evident that any result smaller than that is too small.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

I. Solution by the PROPOSER.

It is well known that every triangle consists of six parts, three sides and three angles, and one side with any two other parts determines the triangle.

In constructing this triangle we may use all possible values, first, of the included angle C, second, of the third side, c, third, of the angle A, and fourth, of the angle B. This gives us four cases.

I. Put angle $C=\theta$. Then $A_1=\dfrac{ab}{2}\displaystyle\int_0^\pi \sin\theta d\theta\div\int_0^\pi d\theta=\frac{ab}{\pi}$.

II. Put side $c=x$. Then $A_2=\frac{1}{4}\displaystyle\int_{a-b}^{a+b}[(a+b)^2-x^2]^{\frac{1}{2}}[x^2-(a-b)^2]^{\frac{1}{2}}dx$.

$$\div\int_{a-b}^{a+b}dx=\frac{a+b}{12b}\left\{(a^2+b^2)E\left[\left(\frac{2\sqrt{ab}}{a-b}\right),\tfrac{1}{2}\pi\right]-(a-b)^2F\left[\left(\frac{2\sqrt{ab}}{a+b}\right),\tfrac{1}{2}\pi\right]\right\}.$$

(To integrate this expression put $x=[(a+b)^2-4ab\sin\theta]^{\frac{1}{2}}$).

III. Put angle $A=\theta$, b being $<a$, then

$$A_3=\tfrac{1}{2}b\int_0^\pi[b\cos\theta+(a^2-b^2\sin^2\theta)^{\frac{1}{2}}]\sin\theta d\theta\div\int_0^\pi d\theta=\frac{ab}{2\pi}+\left(\frac{a^2-b^2}{4\pi}\right)\log_e\left(\frac{a+b}{a-b}\right).$$

IV. Put angle $B=\theta$. For every value of B there are two triangles whose average arc is $\frac{1}{2}a^2\sin\theta\cos\theta$. Hence,

$$A_4=\tfrac{1}{2}a^2\int_0^{\sin^{-1}\frac{b}{a}}\sin\theta\cos\theta d\theta\div\int_0^{\sin^{-1}\frac{b}{a}}d\theta=\tfrac{1}{4}b^2\div\sin^{-1}\frac{b}{a}.$$

COROLLARY. If $b=a$, $A_1=a^2/\pi$, and $A^2=a^2/3$. These are double the values found in the solutions of problem 26, as they evidently should be. The values of A_3 and A_4 do not hold when $b=a$, for the reason that while the sum of the areas remains the same the number of triangles is reduced one-half at the moment that $b=a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let x=third side, A=area, Δ=average area.

$$\therefore\ A=\tfrac{1}{4}\sqrt{(a+b)^2-x^2}\sqrt{x^2-(a-b)^2}.$$

$$\Delta=\int_{a-b}^{a+b}Adx\div\int_{a-b}^{a+b}dx=\frac{1}{2b}\int_{a-b}^{a+b}Adx.$$

Let $(a+b)^2-x^2=4aby^2$, $\dfrac{4ab}{(a+b)^2}=e^2$.

$$\therefore\ \Delta=\frac{2a^2b}{a+b}\int_0^1\frac{y^2\sqrt{1-y^2}dy}{\sqrt{1-e^2y^2}}=\tfrac{1}{2}a(a+b)\int_0^1\frac{y^2\sqrt{1-e^2y^2}dy}{\sqrt{1-y^2}}$$

$$-\frac{a(a-b)^2}{2(a+b)}\int_0^1\frac{y^2dy}{\sqrt{1-y^2}\sqrt{1-e^2y^2}}=\frac{a+b}{12b}\{(a^2+b^2)E(e)-(a-b)^2F(e)\}.$$

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C., and A. P. REED, Clarence, Missouri.

The area of triangle is $\triangle=\frac{1}{2}ab\sin\theta$.

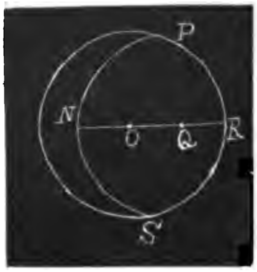
$$\text{Hence, average area}=\tfrac{1}{2}ab\int_0^\pi\sin\theta d\theta\div\int_0^\pi d\theta=\tfrac{1}{2}ab\Big[-\cos\theta\Big]_0^\pi\div\pi=\frac{ab}{\pi}.$$

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two arrows are sticking in a circular target: show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 335.]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Let Q be the position of one arrow. Call the radius of target R, and let $OQ=\rho$.



The area $PNSR=2R^2\cos^{-1}\frac{\rho}{2R}-\rho\sqrt{R^2-(\frac{1}{2}\rho)^2}$.

Then the chance that the second arrow is within the above region is

$$\frac{2}{\pi}\cos^{-1}\frac{\rho}{2R}-\frac{\rho}{\pi R^2}\sqrt{R^2-(\tfrac{1}{2}\rho)^2}.$$

The chance that the first arrow is at a distance ρ from the center is

$$\frac{2\pi\rho d\rho}{\pi R^2}=\frac{2\rho d\rho}{R^2}.$$

The chance that the *two* arrows are as indicated above is

$$\frac{4}{\pi R^2}\cos^{-1}\frac{\rho}{2R}.\rho d\rho-\frac{2}{\pi R^4}\sqrt{R^2-(\tfrac{1}{2}\rho)^2}\,\rho^2 d\rho.$$

The sum of all such chances is

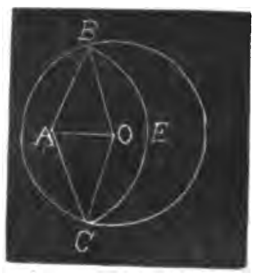
$$\frac{4}{\pi R^2}\int_0^R \rho\cos^{-1}\frac{\rho}{2R}d\rho-\frac{2}{\pi R^4}\int_0^R\sqrt{R^2-(\tfrac{1}{2}\rho)^2}\,\rho^2 d\rho=1-\frac{3\sqrt{3}}{4\pi}.$$

$\therefore$ Chance that the second arrow is *without* the region $PNSR$ is

$$1-\left(1-\frac{3\sqrt{3}}{4\pi}\right)=\frac{3\sqrt{3}}{4\pi}.$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let O be the center of the target and A the position of one of the arrows. Then if the distance between the arrows is greater than the radius, a, the other arrow must lie outside the arc BEC drawn from A as center and radius a. If $AO=x$, the area of the surface outside of the arc BEC is



$$S=2a^2\sin^{-1}\left(\frac{x}{2a}\right)+\tfrac{1}{2}x(4a^2-x^2)^{\frac{1}{2}}.$$

The probability that the one arrow is at the distance x from the center is $2\pi xdx\div a^2\pi=2xdx\div a^2$. The probability that the

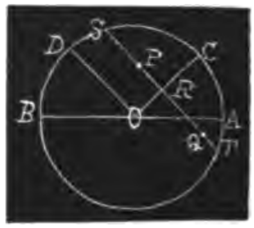
other is on the surface outside the arc BEC is $S\div a^2\pi$. Hence the required probability is

$$P=\frac{2}{a^4\pi}\int_0^a Sxdx. \quad \text{Put } x=2a\sin\theta.$$

$$\text{Then } P=\frac{16}{\pi}\int_0^{\frac{1}{2}\pi}(\theta+\sin\theta\cos\theta)\sin\theta\cos\theta=3\sqrt{3}/4\pi.$$

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P, Q be the arrows, $SQ=x$, $PQ=y$, $ST=u$, $OR=z$, $\angle DOB=\theta$, $OA=a$.



An element of area at Q is $dzdx$; at P, $yd\theta dy$.

The limits of x and 0 are $u-a$; of y, $u-x$ and a, and doubled ; of z, 0 and $\frac{1}{2}a\sqrt{3}$, and doubled ; of θ, 0 and $\frac{1}{2}\pi$, and doubled. Δ=chance, $u=2\sqrt{a^2-z^2}$.

$$\therefore \Delta=\frac{8}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\int_a^{u-x}d\theta dzydxdy$$

$$=\frac{4}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\{(u-x)^2-a^2\}d\theta dzdx$$

$$=\frac{4}{3\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}(u^3+2a^3-3a^2u)d\theta dz=\frac{3\sqrt{3}}{2\pi^2}\int_0^{\frac{1}{2}\pi}d\theta=\frac{3\sqrt{3}}{4\pi}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by THOS. U. TAYLOR, C. E., M. C.; Department of Engineering, University of Texas, Austin, Texas.

Given a variable parallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and (C)$=h$, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by AP+area generated by BC).

Solution by the PROPOSER.

Let (A)=area generated by PA; (B)=area curve generated by B; (C) =area curve generated by C.

Project area (A) orthogonally on plane of (B) and (C). Then by Elliott's Extension of Holditch's Theorem

$$S=x(A)+y(B)-xy(C).$$

where $x+y=1$, and x and y are the radii in which the section S divides the generator. Make $x=y=\frac{1}{2}$.

$$\therefore S_{\frac{1}{2}}=\tfrac{1}{2}(A)+\tfrac{1}{2}(B)-\tfrac{1}{4}(C).$$

But by Newton's formula, V=volume of whole solid

$$=\tfrac{1}{6}H\{(A)+4S_{\frac{1}{2}}+(B)\}=H\{\tfrac{1}{2}[(B)+(A)]-\tfrac{1}{6}(C)\}.$$

Volume of cone$=\frac{1}{3}H(C)$. $\therefore$ Volume generated by

$$APCB=\tfrac{1}{2}H\{(A)+[(B)-(C)]\}=\tfrac{1}{2}H\{\text{area } AP+\text{area } BC\}.$$

34. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minntes, —30 degrees, 12 minutes; of the latter, 16 hours, 23 minutes, —26 degrees, 12 minutes?

II. Solution (continued) by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$h=130^\circ\ 4'\ 57''$ for upper meridian.

$\therefore 180^\circ-130^\circ\ 4'\ 57''=49^\circ\ 55'\ 3''=h$, for lower meridian.

$\therefore h=3$ hours, 19 minutes, 40.2 seconds.

$\therefore$ sidereal time for equal altitudes on latitude 40° south is $a-h=19$ hours, 32 minutes, 19.8 seconds.

$a-h-12=7$ hours, 32 minutes, 19.8 seconds is sidereal time on upper meridian at same moment.

Dr. S. Hart Wright communicated to me the following rather startling discovery which is probably responsible for the problem: The arc of a great circle passing to and between the two stars actually passed through the Nadir. Now when the stars are of equal altitudes they are equally distant from the Nadir as well as from the Zenith.

$\therefore$ The arc between them$=82^\circ\ 51'\ 52.5''$ must be bisected, each being $41^\circ\ 25'\ 56\frac{1}{4}''$.

These facts, if they had been stated, would have made the problem quite simple.

See problem and solution in August-September number.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

71. Proposed by J. C. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6% ?

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-8 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth ? *Encyclopedia Brittanica.*

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

I would like to change problem 70, Arithmetic, to read as follows and have it proposed for solution:

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and *the interest to draw interest until the time of settlement ?*

Solve by simple arithmetic without the aid of algebraic symbols.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at 108¾ and in Philadelphia at 112¼, exchange \$4.89½, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia ?

ALGEBRA.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve according to the conditions given :

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}.$$

First, square without transposing and then solve; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

75. Proposed by B. F. BURLESON, Oneida Castle, New York.

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumscriptible, and contains an area of $k=10752$ square rods. The square described on the radius of its inscribed circle contains $r^2=2304$ square rods; while the square described on the radius of its circumscribed circle contains an area of $R^2=7345$ square rods. Required the lengths of the sides of his farm.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

77. **Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.**

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

GEOMETRY.

69. **Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.**

To divide a square card into right-lined sections in a manner, that a rectangle of a given breadth can be formed from the sections; likewise, form a square from a rectangular card.

70. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.**

Prove that the locus of the center of the circle which passes through the vertex of a parabola and through its intersections with a normal chord is the parabola $2y^2=ax-a^2$, the equation to the given parabola being $y^2=4ax$.

71. Prove by pure geometry: A perpendicular at the middle point, M_a, of the side BC of the triangle ABC meets the circumcircle in A'. On this perpendicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the orthocenter of triangle ABC). Prove that A''' is on the circumcircle. *Anon.*

72. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.**

If a line with its extremities upon two curves move in any manner whatever, (the line may vary in length), and P a point upon the line which divides it in the ratio $m:n$ describe a curve, the area of this curve will be given by the formula—

$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

73. Prove by pure geometry: (1) A', B', and C' are the middle points of the arcs BC, CA, and AB respectively. With these points as centers, circles are described passing through B and C, C and A, and A and B respectively. Prove that these circles intersect in O, the center of the incircle of the triangle ABC; (2), that O, the center of the incircle, is Nagel's point of the triangle formed by joining the middle points of the sides. *Anonymous.*

CALCULUS.

61. **Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.**

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or $2n$ loops according as n is an odd or an even integer; (2) that its area is $\frac{1}{4}$ or $\frac{1}{2}$ of the circumscribing circle according as n is an add or an even integer.

62. **Proposed by A. H. HOLMES, Brunswick, Maine.**

A bucket is in the form of a frustum of a cone having its smaller end as a base. It is a inches in diameter at base and b inches in diameter at top, and its perpendicular

height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket.

63. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

What is the volume removed by boring an auger hole radius R through a right cylinder radius R, the center of the auger hole to pass at a distance c from the axis of the cylinder and inclined to the axis at an angle α ?

64. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, High School, Cleveland, Ohio.

Find volume and surface generated by revolving about y, the catenary $y=\frac{1}{2}a(e^{x\backslash a}+e^{-x\backslash a})$, from $x=0$ to $x=a$. [*Osborne's Calculus*, page 255, example 8.]

MECHANICS.

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, Texas.

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

49. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground.

50. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

A plane quadrilateral $ABCD$ in the vertical wall of a cistern, filled with water, has its four vertices A, B, C, D at the distances 10 feet, 4 feet, 5 feet, and 7 feet respectively, from the surface of the water. The projections of AB, BC, and CD upon the surface are respectively 2 feet, 3 feet, and 1 foot. Find the pressure of the water upon the quadrilateral, and the position of the center of mean pressure.

51. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"Swift of foot was Hiawatha.
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen." *Longfellow.*

Assuming Hiawatha to have been able to shoot an arrow every second and to have aimed when not shooting vertically so that the arrow might have the longest range; what was Hiawatha's time in a hundred yards ?

AVERAGE AND PROBABILITY.

47. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of the chords that may be drawn from one extremity of the major axis of an ellipse to every point of the curve ?

48. Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation ?

49. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fastened together at their centers, but otherwise free to move. If they are thrown on a floor at random, what is the average area common to both?

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find (1), the average length of all straight lines having a given direction, between 0 and a; (2), the average length of chords drawn from one extremity of the diameter a of a semi-circle to all points in the semi-circumference; and (3), find the average area of all triangles formed by a straight line of constant length a sliding between two straight lines at right angles.

[Solutions of these problems should be sent to the editors of the respective departments on or before February 1, 1897.]

EDITORIALS.

Our valued contributor, Prof. O. W. Anthony, has been elected Professor of Mathematics in the Columbian University, Washington, D. C.

James F. Lawrence, I. F. Yothers, G. B. M. Zerr, J. C. Corbin, Frederick R. Honey, H. C. Wilkes, and Nelson S. Roray should have received credit for solving Nos. 66, 67, 68, and 69, Department of Arithmetic. O. W. Anthony should have received credit for solving No. 63, Department of Geometry. We wish to state again that all solutions, to receive credit, should be sent to the proper editor; but this remark does not apply to the above persons.

The MONTHLY will soon begin its fourth volume. Will not every one of its old subscribers try and secure one new subscriber for the coming year? Send us names of persons likely to subscribe and we shall take pleasure in sending them sample copies. Persons sending us three new subscribers and remitting us $6.00 will receive a years subscription as a premium.

Some of our readers have suggested that we publish in groups portraits of our contributors. If this suggestion meets the approval of our contributors, we shall be pleased to receive photos which we will have grouped by one of the best artists in Springfield, and shall furnish the plates at cost to us. We shall be pleased to hear from the contributors to the MONTHLY in reference to this matter.

A letter from Dr. Halsted dated November 27th, 1896, says, "For four months I was buried in the uttermost parts of Hungary, Russia, and Siberia, and am just getting used to English again. I made many important finds and had many strange experiences." There are few other Americans whose travels in Russia would have been as important to the Non-Euclidean Geometry as this trip of Dr. Halsted's. He is already working on some very important translations which will soon be made known for the first time to English speaking mathematicians.

BOOKS AND PERIODICALS.

Elements of Mechanics, Including Kenematics, Kinetics, and Statics, with Applications. By Thomas Wallace Wright, M. A., Ph. D., Professor in Union College. 8vo. Cloth, 372 pages. Price, $2.50. New York: D. Van Nostrand Company.

This is a completely rewritten edition of the author's Text-book of Mechanics. The same general plan has been followed, but many changes in detail have made, so the book comes before the public with a new name. In this book much use is made of the graphical method; machines are discussed in great detail; the important subjects of oscillation and rotation have been treated with more fullness than is usual in an elementary treatise. Numerous well chosen problems are appended to the discussion, while at the end of each chapter is added a series of examination questions. Historical notes are freely interspersed to add a more live interest to the subject. This is a very excellent book and we very heartily recommend it to teachers desiring a good work on Mechanics. B. F. F.

The Elements of Physics. A College Text-book. By Edward L. Nichols and William S. Franklin. In three volumes, Vol. II. Electricity and Magnetism. 8vo. Cloth, ix and 272 pages. Price, $1.50. New York: The Macmillan Co.

In the study of this excellent work a knowledge of the elementary principles of the calculus and quaternions is required. This fact will preclude its use in many colleges. The authors recognizing, however, that there is a growing tendency among the best colleges to increase the requirements in mathematics, these colleges realizing that the discipline received from the study of mathematics is not excelled by any other branch of study, have not slurred over certain parts of Physics containing *real and unavoidable difficulties*. Nor have those portions containing these difficulties been omitted, but they have been faced frankly; the statements involving them having been reduced to the simplest form which is compatible with accuracy. Colleges in which only one course is offered in Physics should at once so adjust their courses of study as to make it possible to use a text-book such as the one before us, as a course of Physics pursued in accordance with the plan of this work will be of infinitely more value both from a practical and an educational point of view, than two or three popular courses requiring only a knowledge of Elementary Algebra and Geometry. B. F. F.

Elements of Plane and Spherical Trigonometry. By C. W. Crockett, Professor of Mathematics and Astronomy, Renssellaer Polytechnic Institute, Troy, New York. Large 8vo. Cloth, 192 pages and 120 pages of tables. Price, $1.25. New York and Chicago: American Book Company.

This work is fully up to the standard of good text-books. It contains a full course in Plane and Spherical Trigonometry; in fact, all that is needed in a course in the best schools and colleges. There are many examples and illustrations. The typographical and mechanical execution of the work is first-class. B. F. F.

Darwinism and Non-Euclidean Geometry. Reprint from the Bulletin de La Société Physico-Mathématique de Kasan. Tome VI. No. 3—4. By Dr. George Bruce Halsted. Pamphlet, 4 pages.

This interesting article seems to have been written by Dr. Halsted while visiting at Kasan in July and August of last summer. In his travels he explored many libraries and made many important finds. B. F. F.

The Maine Farmer's Almanac for 1897.

Through the courtesy of Prof. William Hoover, of Athens, Ohio, we received a copy of this noted little Almanac, which, among other important and useful information, contains two pages devoted to Mathematical Questions and Solutions. The price of the Almanac is 10 cents. B. F. F.

Prismoidal Formulae, with Special Derivation of Two-Term Formulae. By Thomas U. Taylor, C. E. (University of Virginia), M. C. E. (Cornell), Associate Professor of Applied Mathematics, University of Texas. Pamphlet, 55 pages.

This paper, which was read before the Texas Academy of Science, March, 1896, adds some valuable material to the literature of Prismoidal Formulae. B. F. F.

Mathematical Questions and Solutions. From the "Educational Times," with an Appendix. Edited by W. J. C. Miller, B. A. Vol. LXV., 8vo. Boards, 128 pages. Francis Hodgson, 89 Farringdon Street, E. C., London.

This valuable reprint contains solutions of about 165 problems. Our readers who secure it will find many interesting problems with their solutions. The price is 5s., 3d., postpaid. J. M. C.

Elementary Hydro-Statics. University Tutorial Series. By William Briggs and G. H. Bryan. Cloth, 208 pages. Price, 50 cents. New York: W. B. Clive, 65 Fifth Avenue.

This work is written in a suggestive and attractive manner. In scope and in method it is admirably adapted to class use as an elementary text. In the examples results are deduced from first principles, and thus the student is not lead to rely on memory for his formulae. The new features are good, the examples are numerous and well selected, and the topical index convenient and useful. J. M. C.

Inductive Manual of Straight Line and Circle. By William J. Meyers, Professor of Mathematics, State Agricultural College of Colorado. Published by the Author, Fort Collins, Colorado, 1896. 113 pages. Price, 60 cents.

The fundamental idea of the book seems to be to furnish the student the tools and material, and by the aid of helpful questions where needed, to have him work up his ideas for himself, in all cases leaving some actual work and thought to the student himself. As distinguishing features we notice: A constant effort to keep prominent the connection between geometrical relations and their applications in the arts; the early introduction and use of the notions of locus and of symmetry; distinction between the obverse and reverse of plane figures; and the closeness of relation between regular chains, polygons, and the circle. There are numerous exercises and problems. It must be left to actual trial to determine its adaptation to class use. J. M. C.

The Alumni Bulletin of the University of Virginia, for November, contains an appreciative sketch, with portrait, of our esteemed subscriber, Professor Charles Scott Venable, LL. D., who lately retired from the head professorship of mathematics at the University of Virginia, a position he has held for over thirty years. J. M. C.

We have received the following valuable papers, in pamphlet form, from Dr. Artemas Martin, editor of the *Mathematical Magazine:* "About Cube Numbers whose Sum is a Cube Number"; About Biquadrate Numbers whose Sum is a Biquadrate Number"; Notes about Square Numbers whose Sum is either a Square or the Sum of other Squares"; On Fifth-Power Numbers whose Sum is a Fifth Power"; and "Solutions of the 'Duck' Problem." Those interested in the subjects of which these papers treat cannot afford to miss them.

The last number of the *Magazine*, issued in May, 1896, contains the paper on Biquadrate Numbers, and the second installment of that on Cube Numbers. Three interesting problems are solved and ten new ones are proposed. J. M. C.

The following periodicals have been received: Journal de Mathématiques Elémentaires, (1er December, 1896); American Journal of Mathematics, (October, 1896); The Mathematical Gazette, (October, 1896); L' Intermédiare des Mathématiciens, (November, 1896); Miscellaneous Notes and Queries, (December); The Kansas University Quarterly, (October, 1896), The Monist, (October, 1896); Bulletin of the American Mathematical Society, (December, 1896); The Educational Times, (November, 1896); The Mathematical Review, (July, 1896); The Mathematical Magazine, (No. 10, issued in May, 1896); Annals of Mathematics, (September, 1896). J. M. C.

THE AMERICAN MATHEMATICAL MONTHLY's Clubbing List:

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Other magazines not mentioned above may be obtained from us at reduced rates.

B. F. FINKEL and J. M. COLAW, Editors.

ERRATA.

On page 221, for "numbers" read *terms* in line 16.

In solution of problem 42, page 220, the part under Example 2, reading, "For $p-q$, $a=9/2$, $b=13/2$, etc.," should be under Example 1, to tally with "for $p+q$, etc."

Page 234, line 5, for "ρ^{-1}" read $\rho^{-\frac{1}{2}}$.

Page 234, line 5, for $\sqrt{\rho+\rho}$" read $\rho+\sqrt{\rho}$.

Page 243, line 5, omit decimal point in denominator.

Page 258, in Figure, read D for "B" and B for "D".

Page 259, multiply the numerator of the right hand member in the value of p by 2.

Page 288, problem 38, the figure is wrong. The arc CE should be *parallel* to BA, as the solution says. Also, BC, which is an arc of the horizon, should be in a level plane.

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SOLUTIONS AND PROBLEMS.

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∴ The numbers in parenthesis refer to the pages where the problems, solutions, and articles are found.

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FINKEL, B. F., A. M., Author of Finkel's Mathematical Solution Book, Member of the American Mathematical Society, Co-Editor of THE AMERICAN MATHEMATICAL MONTHLY, and Professor of Mathematics and Physics in Drury College, Springfield, Mo. (55, 59, 115, 118, 139, 144, 150, 153, 155, 159, 187, 191, 207, 219, 239, 256, 259, 285, 319, 330)

FISH, EDMUND, Hillsboro, Ill. (189)

FRAKER, L. B., Weston, O. (113, 139)

GEARTNER, Prof. H. J., Wilmington, O. (118, 119)

GREGG, J. C., A. M., Superintendent of City Schools, Brazil, Ind. (118, 119, 120, 314, 316)

GRUBER, M. A., A. M., War Department, Washington, D. C. (52, 56, 80, 81. 151, 152, 153, 177, 212, 220, 221, 274, 278, 282, 284)

HALSTED, GEORGE BRUCE, A. M., Ph. D., Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Tex. (1-5, 13-14, 24, 25, 35, 67, 91, 109, 122, 132, 297)

HEAL, WILLIAM E., A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Ind. (41, 92, 181, 216, 277)

HEATON, HENRY, M. Sc., Atlantic, Iowa. (139, 154, 178, 192, 210, 211, 236, 242, 257, 259, 288, 302, 314, 323, 325, 330)

HILLSBORO MATHEMATICAL CLUB, Hillsboro, Ill. (90-91)

HOBBS, CHAS. A., A. M., Master of Mathematics in the Belmont School, Belmont, Mass. (213)

HOLDEN, WARREN, Professor of Mathematics, Girard College, Philadelphia, Pa. (123, 291)

HOLMES, A. H., Box 963, Brunswick, Maine. (145, 146, 148, 155, 177, 188, 189, 212, 221, 222, 226, 244, 250, 313, 329)

HONEY, FREDERICK R., Ph. B., New Haven, Conn. (57, 118, 145, 177, 180, 238, 275, 302)

HOOVER, WILLIAM, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, O. (119, 146, 154, 181, 213, 214, 239, 241, 242, 276, 280, 329)

HOPKINS, Prof. G. I., Instructor in Mathematics and Physics in High School, Manchester, N. H. (118, 329)

HUGHLETT, A. M., M. A., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Va. (246, 249)

HUME, ALFRED, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Miss. (21, 52, 216, 235, 278)

JOHNSON, EDGAR H., Professor of Mathematics, Emory College, Oxford, Georgia. (275)

KESNER, EDGAR, Boulder, Col. (52, 57, 80)

KING, W. F., Ottawa, Canada. (221, 319)

LAWRENCE, JAMES F., Breckenridge, Mo. (302)

LANDIS, W. W., A. M., Dickenson College, Carlisle, Pa. (313)

LOOMIS, E. S., Ph. D., Cleveland, O. (330)

LILLEY, GEORGE, Ph. D., LL. D., 394 Hall Street, Portland, Ore. (90, 119, 148, 184, 308)

LYLE, JOHN N., Ph. D., Bentonville, Ark. (77, 92, 269)

MAHONEY, J. OWEN, B. E., M. Sc., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tenn. (118, 119, 181, 214, 216, 313)

MARTIN, ARTEMAS, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C. (60, 83, 151, 223)

MATZ, F. P., D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa. (21, 22, 55, 56, 58, 84, 85, 88, 95, 113, 140, 144, 152, 154, 176, 179, 181, 186, 187, 190, 191, 192, 209, 210, 226, 238, 239, 250, 258, 259, 275, 277)

McGAW, F. M., A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, N. J. (140, 176, 185, 208, 209, 240, 307)

MILLER, G. A., Ph. D., Liepzig, Germany. (7, 36, 38, 69, 104, 133, 171, 197, 295, 313)

MOORE, E. H., Ph. D., Professor of Mathematics in the University of Chicago, Chicago, Ill. (38)

MORITZ, ROBT. E., Professor of Mathematics in Hastings College, Hastings, Neb. (52)

MORRELL, E. W., Professor of Mathematics in Montpelier Seminary, Montpelier, Vt. (51, 80, 116, 118, 119, 211, 212, 221, 289. 290, 301)

MYERS, CHARLES E., Canton, O. (115, 259)

NAGLE, J. C., A. M., M. C. E., Professor of Civil Engineering in the State A. and M. College, College Station, Texas. (315)

NEIKIRK, LEWIS, Senior in the University of Colorado, Boulder, Col. (286, 287)

PHILBRICK, P. H., M. S., C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Pineville, La. (265, 330)

PRATT, SETH, C. E., Assyria, Mich. (89, 256, 290)

PRIEST, F. M., Mona House, St. Louis, Mo. (156, 278)

READ, A. P., A. M., Clarence, Mo. (53, 56, 140, 176, 209, 243)

ROBRINS, EDWABD R., Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, N. J. (176, 208, 209, 226, 238, 248, 249, 274, 278)

RORAY, NELSON S., South Jersey Institute, Bridgeton, N. J. (328)

ROBBINS, SYLVESTER, North Branch Depot, N. J. (189)

ROSS, ALEXANDER, C. E., Sebastopol, Cal. (155)

SCHEFFER, Prof. J., A. M., Hagerstown, Md. (52, 53, 80, 81, 118, 119, 120, 122, 146, 148, 151, 152, 176, 181, 209, 242, 245, 250, 289, 290, 304, 305, 313, 330)

SCHMITT, COOPER D., M. A., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn. (52, 53, 80, 81, 82, 117, 119, 141, 142, 152, 157, 177, 189, 212, 214, 221)

SCHWATT, I. J., Ph. D., Professor of Mathematics in the University of Pennsylvania, Philadelphia, Pa. (17, 57, 145, 179, 239, 242)

SHERWOOD, E. L., A. M., Principal of City Schools, West Point, Miss. (57, 80, 121, 146, 147, 184, 187)

SHIELDS, F. M., County Surveyor, Coopwood, Miss. (54, 225, 248, 291)

SINE, Prof. B. F., Principal of High School, Rock Enon Springs, Va. (118, 119)

SMITH, ALWIN C., The University of Colorado, Boulder, Col. (285)

SMITH, DAVID EUGENE, Ph. D., Professor of Mathematics in the Michigan State Normal School, Ypsilanti, Mich. (29, 60, 142)

SMITH, O. D., A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Ala. (122)

SMITH, WILLIAM BENJAMIN, Ph. D., Professor of Mathematics, Tulane University, New Orleans, La. (163)

STEVENS, MOSES C., A. M., Department of Mathematics, Purdue University, Lafayette, Ind. (256, 282, 314, 329)

SYMMONDS, WILLIAM, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O., Sebastopol, Cal. (22, 157, 192, 224)

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WATSON, J. W., Middle Creek, O. (213)

WHITAKER, H. C., M. E., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pa. (20, 90, 121, 140, 176, 181, 244, 281, 289, 314, 330)

WHITE, Prof. C. E., A. M., Trafalgar, Ind. (21, 143, 210)

WILKES, H. C., Skull Run, W. Va. (52, 54, 82, 113, 114, 140, 151, 174, 175, 177, 188, 194, 209, 213, 221, 222, 249, 283, 284, 305)

WILLIAMS, J. C., Rome, N. Y. (56, 288)

WIREBACK, I. J., M. D., St. Petersburg, Pa. (291)

WOOD, DeVOLSON, M. A., C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, N. J. (21)

WRIGHT, SAMUEL HART, M. D., M. A., Ph. D., Penn Yan, N. Y. (88, 89, 226, 288)

YANNEY, BENJ. F., A. M., Professor of Mathematics in Mount Union College, Alliance, O. (50, 52, 65, 80, 81, 82, 89, 90, 110, 113, 114, 116, 118, 141, 142, 146, 148, 152, 153, 169, 177, 179, 193, 211, 212, 221, 226, 299)

YOTHERS, J. F., Westerville, O. (238)

YOUNG, R. H., West Sunbury, Pa. (80, 82)

ZERR, Prof. G. B. M., A. M., Ph. D., Texarkana, Ark.-Tex. (17, 20, 21, 46, 52, 53, 54, 57, 58, 73, 80, 81, 82, 85, 88, 100, 116, 118, 120, 122, 140, 141, 144, 145, 146, 147, 148, 152, 154, 155, 174, 175, 177, 178, 179, 181, 183, 185, 186, 188, 191, 192, 193, 203, 209, 211, 214, 216, 217, 221, 224, 225, 232, 238, 240, 242, 243, 244, 247, 250, 251, 254, 258, 259, 274, 275, 276, 277, 278, 279, 282, 284, 287, 289, 290, 302, 303, 305, 310, 313, 315, 317, 324, 326, 327, 330)

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